BM $\{W_t\}$, $\langle W, W\rangle_t = t$.

e.g.: $\{W_t\}$ BM, $h = \frac{t}{n}$,

$$Z_n = \sum_{j=0}^{n-1} \left(W_{(j+1)h} - W_{jh}\right)^2, \text{ prove } Z_n \xrightarrow{L^2} t \quad (n \to \infty).$$

Pf:

$$\mathbb{E}(Z_n - t)^2 = \mathbb{E}\left[\sum_{j=0}^{n-1} \left(W_{(j+1)h} - W_{jh}\right)^2 - \underbrace{\boxed{t}}_{=\sum_{j=0}^{n-1}((j+1)h - jh)}\right]^2$$

$$= \mathbb{E}\left(\sum_{j=0}^{n-1} \left[\left(W_{(j+1)h} - W_{jh}\right)^2 - h\right]\right)^2$$

$$= \mathbb{E} \sum_{i,j=0}^{n-1} \left[\left(W_{(j+1)h} - W_{jh}\right)^2 - h\right] \cdot \left[\left(W_{(i+1)h} - W_{ih}\right)^2 - h\right]$$

$$= \sum_{i,j=0}^{n-1} \mathbb{E}\left[\left(W_{(j+1)h} - W_{jh}\right)^2 - h\right] \cdot \left[\left(W_{(i+1)h} - W_{ih}\right)^2 - h\right]$$

$$= \underbrace{\sum_{i=1}^{n} \mathbb{E}\left[\left(W_{(i+1)h} - W_{ih}\right)^2 - h\right]^2}_{i=j} +$$

$$\underbrace{2 \sum_{i<j} \mathbb{E}\left[\left(W_{(j+1)h} - W_{jh}\right)^2 - h\right] \cdot \mathbb{E}\left[\left(W_{(i+1)h} - W_{ih}\right)^2 - h\right]}_{\boxed{i \neq j}}$$

independent.

constantly 0.

$$\mathbb{E}\left(\underbrace{W_{(j+1)h} - W_{jh}}_{\sim N(0,h)}\right)^2 - h = 0.$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[\left(W_{(i+1)h} - W_{ih}\right)^2 - h\right]^2$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[\left(\underbrace{W_{(i+1)h} - W_{ih}}_{\sim N(0,h)}\right)^4 - 2h \cdot \left(\underline{W_{(i+1)h} - W_{ih}}\right)^2 + h^2\right]$$

if $G \sim N(0,h)$

$\mathbb{E}G^4 = 3 \cdot h^2$

$$= \sum_{i=1}^{n} \left[3h^2 - 2h \cdot h + h^2\right] = 2h^2 \cdot n = 2 \cdot h \cdot \boxed{t}$$

$\longrightarrow 0 \quad (n \to \infty)$

if we take $n \to \infty$, $h \to 0$

Lebesgue-Stieljes integral: w.r.t. a trajectory with finite variation

$$\int f(s)\,ds, \quad \int f(s)\,d\underbrace{F(s)}_{\text{CDF.}} \quad \text{(expectation)}$$

$$\int f(s)\,d\underbrace{G(s)}_{\text{finite variation}}$$

trajectory of SP has high perturbation

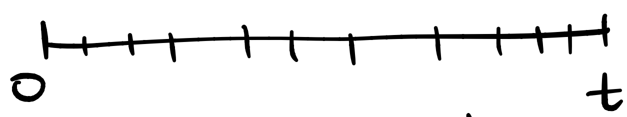
$$\langle W, W\rangle_t = t \neq 0$$

non-zero quadratic variation $\Rightarrow$ infinite total variation

Stochastic integral: $\int_0^t s\,\underbrace{dW_s}_{\text{integrate w.r.t. trajectory with inf total variation}}$,

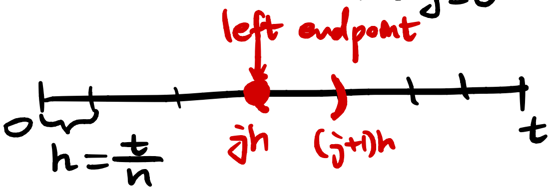
$$\int_0^t W_s\,dW_s$$

Ito integral:



use left-endpoint!

e.g.: $\int_0^t W_s\,dW_s = \lim_{n\to\infty}\sum_{j=0}^{n-1} W_{\boxed{jh}}\left(W_{(j+1)h} - W_{jh}\right)$



(a): left endpoint

$$\sum_{j=0}^{n-1} W_{\boxed{jh}}\underbrace{\left(W_{(j+1)h} - W_{jh}\right)}_{\sim N(0,h)\ \text{independent of}\ W_{jh}} = I_1(n)$$

$$\|$$

$$\frac{1}{2}\left(\sum_{j=0}^{n-1}\left(W^2_{(j+1)h} - W^2_{jh}\right) - \sum_{j=0}^{n-1}\left(W_{(j+1)h} - W_{jh}\right)^2\right)$$

$$\cancel{W^2_{(j+1)h}} - W^2_{jh} - \cancel{W^2_{(j+1)h}} + 2W_{(j+1)h}W_{jh} - W^2_{jh}$$

$$= 2W_{jh}\left(W_{(j+1)h} - W_{jh}\right)$$

$$\downarrow$$

$$\frac{1}{2}\left(W_t^2 - \underset{\langle W,W\rangle_t}{\underset{\|}{\boxed{t}}}\right) \quad \text{as } n\to\infty.$$

$$\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 \boxed{-\frac{1}{2}t}$$

Ito correction term, tells us that Ito integral has no chain rule!

$$\boxed{\int_0^t s\,ds = \frac{1}{2}s^2\Big|_{s=0}^{t} = \frac{1}{2}t^2}$$

(b): Right endpoint (denote integral as Δ)

$$\sum_{j=0}^{n-1} W_{(j+1)h}\left(W_{(j+1)h} - W_{jh}\right)$$

$$W_{(j+1)h}^2 - W_{(j+1)h}W_{jh} = \frac{1}{2}\left(W_{(j+1)h} - W_{jh}\right)^2$$

$$-\frac{1}{2}W_{(j+1)h}^2 - \frac{1}{2}W_{jh}^2 + W_{(j+1)h}^2$$

$$= \frac{1}{2}\left(W_{(j+1)h} - W_{jh}\right)^2 + \frac{1}{2}\left(W_{(j+1)h}^2 - W_{jh}^2\right)$$

$$= \frac{1}{2}\sum_{j=0}^{n-1}\left(W_{(j+1)h} - W_{jh}\right)^2 + \frac{1}{2}\sum_{j=0}^{n-1}\left(W_{(j+1)h}^2 - W_{jh}^2\right)$$

$$\longrightarrow \frac{1}{2}W_t^2 + \frac{1}{2}t \qquad (n\to\infty)$$

$$\int_0^t W_s \,\Delta\, dW_s = \frac{1}{2}W_t^2 + \frac{1}{2}t$$

(c): Midpoint rule (denote integral by $\circ$)

$$\sum_{j=0}^{n-1} \frac{W_{jh} + W_{(j+1)h}}{2} \cdot \left(W_{(j+1)h} - W_{jh}\right)$$

$$\longrightarrow \frac{1}{2} \cdot \left(\frac{1}{2}W_t^2 - \frac{1}{2}t\right) + \frac{1}{2} \cdot \left(\frac{1}{2}W_t^2 + \frac{1}{2}t\right)$$

$$= \frac{1}{2}W_t^2 \quad (n \to \infty)$$

Stratonovich integral

$$\boxed{\int_0^t W_s \circ dW_s = \frac{1}{2}W_t^2}$$

satisfy the chain rule!

e.g: Calculate $\int_0^t s\,dW_s$ (Ito integral), $h=\frac{t}{n}$

Pf: $\displaystyle\sum_{j=0}^{n-1} \underline{jh} \cdot \left(W_{(j+1)h} - W_{jh}\right)$

left endpoint.

$$= h \cdot \sum_{j=0}^{n-1} j \left(W_{(j+1)h} - W_{jh}\right)$$

$$= h \cdot \sum_{j=0}^{n-1} \sum_{k=1}^{j} \left(W_{(j+1)h} - W_{jh}\right)$$

$$= h \cdot \sum_{k=1}^{n-1} \underbrace{\sum_{j=k}^{n-1} \left(W_{(j+1)h} - W_{jh}\right)}_{W_t - W_{kh}}$$

$$= h \cdot \left(\underbrace{\sum_{k=1}^{n-1} W_t}_{n \cdot W_t} - \sum_{k=1}^{n-1} W_{kh} \right)$$

$$= t \cdot W_t - \sum_{k=1}^{n-1} h \cdot W_{kh}$$

$$\longrightarrow t \cdot W_t - \int_0^t W_s\,ds \quad (n \to \infty)$$

$$\boxed{\begin{aligned} &\int_0^t s\,dW_s \\ &\quad \parallel \\ &t \cdot W_t - \int_0^t W_s\,ds. \\ &\quad \text{checked.} \end{aligned}}$$

e.g: $X_t = \int_0^t W_s\,ds$, is a Gaussian proc, find mean func, autocovariance.

Pf: $X_t = \lim_{n\to\infty} \sum_{j=0}^{n-1} W_{jh}\cdot h$, GP ✓

$$\mathbb{E}X_t = \mathbb{E}\int_0^t W_s\,ds = \int_0^t \mathbb{E}W_s\,ds = 0.$$

$$\operatorname{cov}(X_s, X_t) = \mathbb{E}X_sX_t \qquad (\forall 0<s<t)$$

$$= \mathbb{E}\int_0^s W_u\,du \int_0^t W_v\,dv$$

$$= \mathbb{E}\int_0^s\int_0^t W_uW_v\,dv\,du$$

$$= \int_0^s\int_0^t \underbrace{\mathbb{E}(W_uW_v)}_{\operatorname{cov}(W_u,W_v) = u\wedge v}\,dv\,du$$

$$= \int_0^s\int_0^t u\wedge v\,dv\,du$$

$$= \int_0^s\underbrace{\int_0^u v\,dv}_{\frac{1}{2}u^2}\,du + \int_0^s\int_u^t u\,dv\,du$$

$$= \int_0^s \tfrac{1}{2}u^2\,du + \int_0^s u(t-u)\,du$$

$$= \frac{1}{6}u^3\Big|_{u=0}^{s} + \left(t\cdot\frac{u^2}{2} - \frac{1}{3}u^3\right)\Big|_{u=0}^{s}$$

$$= \frac{1}{6}s^3 + \frac{s^2t}{2} - \frac{1}{3}s^3$$

$$= \boxed{\frac{s^2t}{2} - \frac{1}{6}s^3}$$

When $s=t$,

$$Var(X_t) = \frac{t^3}{2} - \frac{1}{6}t^3 = \underline{\underline{\frac{1}{3}t^3}}.$$