Markovian Diffusion $\{D_t\}$, cts sample path,

$$\begin{cases} \forall \varepsilon>0, \mathbb{P}(|D_{t+h}-D_t|>\varepsilon \mid D_t=x) = o(h) \\ \mathbb{E}(D_{t+h}-D_t \mid D_t=x) = \underbrace{a(t,x)}_{\text{drift}} h + o(h) \quad (h\to 0) \\ \mathbb{E}[(D_{t+h}-D_t)^2 \mid D_t=x] = \underbrace{b(t,x)}_{\text{diffusion}} h + o(h) \end{cases}$$

Generalization: $dD_t = a(t,D_t)\,dt + b(t,D_t)\,dW_t$

(SDE)

How does $\{D_t\}$ propagate w.r.t. time?

$$\underbrace{f(t,y\mid s,x)}_{\text{conditional pdf}} \triangleq \frac{\partial}{\partial y}\mathbb{P}(D_t\le y \mid D_s=x)$$

conditional pdf

of D_t given $D_s=x$,

y is the space variable

$$\begin{cases} \text{forward eqn (Fokker-Planck)}: \frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}[a(t,y)\cdot f] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[b(t,y) f] \\ \text{backward eqn (Backward Kolmogorov)}: \frac{\partial f}{\partial s} = -a(s,x)\frac{\partial f}{\partial x} - \frac{1}{2}b(s,x)\frac{\partial^2 f}{\partial x^2} \end{cases}$$

$$\boxed{\begin{matrix}\text{SDE}\\ \Updownarrow \\ \text{PDE}\end{matrix}}$$

e.g.: BM, $a(t,x)=0$, $b(t,x)=1$, $W_0=0$

forward eqn: $\begin{cases} \frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \\ f(0,y) = \delta_0(y) \end{cases}$ $(s=0,\ x=0,\ f=f(t,y))$

heat equation,
fundamental solution as Gaussian kernel.

<u>Check</u>: $f(t,y)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}$ is the solution.

$$\frac{\partial f}{\partial t}=\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{y^2}{2t}}+t^{-\frac{1}{2}}\cdot e^{-\frac{y^2}{2t}}\cdot\frac{y^2}{2t^2}\right)$$

$$\frac{\partial f}{\partial y}=\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}\cdot\left(-\frac{y}{t}\right)=\frac{1}{\sqrt{2\pi}}t^{-\frac{3}{2}}(-y)\cdot e^{-\frac{y^2}{2t}}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{1}{\sqrt{2\pi}}t^{-\frac{3}{2}}\cdot\left(-e^{-\frac{y^2}{2t}}+\frac{y^2}{t}e^{-\frac{y^2}{2t}}\right)$$

so $\frac{\partial f}{\partial t}=\frac{1}{2}\frac{\partial^2 f}{\partial y^2}$, $f(0,y)=\lim_{t\to 0}\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}=\delta_0(y)$

the solution f is actually pdf of $N(0,t)$
which matches $W_t \sim N(0,t)$.

e.g.: BM with drift $a(t,x)=m$, $b(t,x)=\sigma^2$,

$X_t = \sigma W_t + mt + d$, so forward eqn:
(d>0)

$$\frac{\partial f}{\partial t} = -m\frac{\partial f}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2}$$

If an absorbing barrier is put up at 0, then it can be reflected in the boundary condition!

$$\begin{cases} f(0,y) = \delta_d(y) \\ f(t,0) = 0 \end{cases}$$

→ if absorbed by 0, it's permanently 0!

Let's solve this PDE with BC:

pdf of $N(x+mt, t)$

first check $g(t,y|x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x-mt)^2}{2t}}$ is the solution to PDE for $\forall x$, but it does not satisfy the BC.

$$\Downarrow$$

$$f(t,y) = g(t,y|d) - e^{-2md} g(t,y|-d)$$

is the solution.

This is called "method of images", which is also the technique used in pricing barrier option, similar to reflection principle.

$T \triangleq$ first absorption time, time t $\begin{cases} \text{absorbed} \\ \text{or} \\ \text{has pdf } f \end{cases}$, so

$$\mathbb{P}(T \leq t) = 1 - \int_0^{+\infty} f(t,y)\,dy$$

$$= 1 - \Phi\left(\frac{d+mt}{\sqrt{t}}\right) + e^{-2md}\,\Phi\left(\frac{-d+mt}{\sqrt{t}}\right)$$

provides cdf of T.

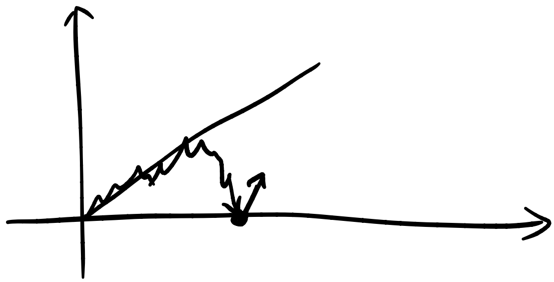
$$\boxed{\mathbb{P}(T<\infty) = e^{-2md} \quad \text{if } m>0}$$

$$\Downarrow$$

$$f_T(t) = \frac{d}{\sqrt{2\pi t^3}}\, e^{-\frac{(d+mt)^2}{2t}} \quad (t \geq 0)$$

What if there's a reflection barrier?

$f(t,y)$ now denotes the law of the reflected process at time t.



Then $\int_0^{+\infty} f(t,y)\,dy = 1$, $\forall t$, $\frac{\partial}{\partial t}$ on both sides,

$$0 = \int_0^{+\infty} \frac{\partial f}{\partial t}\,dy = \int_0^{+\infty} \left(-m\frac{\partial f}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2}\right) dy$$

$$\underset{\text{parts}}{\overset{\text{int by}}{=\!=}} \; m f(t,0) - \frac{\sigma^2}{2}\frac{\partial f}{\partial y}(t,0)$$

So: forward eqn

$$\begin{cases} \dfrac{\partial f}{\partial t} = -m\dfrac{\partial f}{\partial y} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 f}{\partial y^2} \\ f(0,y) = \delta_d(y) \\ m\cdot f(t,0) - \dfrac{\sigma^2}{2}\dfrac{\partial f}{\partial y}(t,0) = 0 \end{cases}$$

how to satisfy this?

Ansatz:

$$f(t,y) = g(t,y|d) + \underbrace{A\cdot g(t,y|-d)}_{\text{reflection}} + \underbrace{\int_{-\infty}^{-d} B(x)\cdot g(t,y|x)\,dx}_{\text{a line of reflections over } (-\infty,-d)}$$

$\Downarrow$

plug into the equation and calculate to get

$$A = e^{-2md}, \quad B(x) = -2m\,e^{2mx}$$

$$\begin{cases} \text{as } t\to\infty,\ f(t,y)\to 0 & \text{if } m\geq 0 \\ \text{as } t\to\infty,\ f(t,y)\to 2|m|\,e^{-2|m|x} & \text{if } m<0 \end{cases}$$

since $\displaystyle\int_{-\infty}^{-d} 2me^{2mx}\, g(t,y|x)\,dx$ $\qquad u = \dfrac{x-y-mt}{\sqrt{t}}$

$$= \int_{-\infty}^{-\frac{d+y+mt}{\sqrt{t}}} 2me^{2my}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{u^2}{2}}\,du$$

$$\longrightarrow \begin{cases} 2|m|e^{-2|m|y} & \text{if } m<0 \\ 0 & \text{if } m\geq 0 \end{cases} \quad \text{as } t\to\infty.$$