13.4.1: $\{W\}$ is std BM, $X_t = e^{i\theta W_t + \frac{1}{2}\theta^2 t}$, show that $\{X_t\}$ is MG w.r.t. $\mathcal{G}_t = \sigma(W_u : u \le t)$

Pf: adapted ✓ $\forall t$, $\mathbb{E}|X_t| \le e^{\frac{1}{2}\theta^2 t} \cdot \mathbb{E}\left|e^{i\theta W_t}\right| < \infty$

integrable ✓

$$\forall\, 0 \le s < t,\quad \mathbb{E}(X_t \mid \mathcal{G}_s) = \mathbb{E}\left(e^{i\theta(W_t - W_s)} \cdot e^{i\theta W_s} \mid \mathcal{G}_s\right) \cdot e^{\frac{1}{2}\theta^2 t}$$

$$= e^{\frac{1}{2}\theta^2 t} \cdot e^{i\theta W_s} \cdot \underline{\mathbb{E}\, e^{i\theta(W_t - W_s)}}$$

$W_t - W_s \sim N(0, t-s)$, by Gaussian c.f., $= e^{-\frac{1}{2}\theta^2(t-s)}$

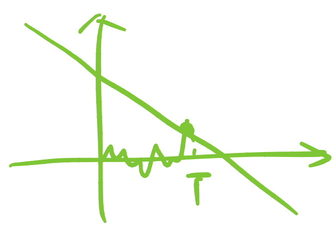
$$= e^{i\theta W_s + \frac{1}{2}\theta^2 s} = X_s \quad ✓$$

exp-MG of BM

13.4.2: $\underline{T \triangleq \inf\{t : W_t = at + b\}}$ $(a<0,\ b\ge 0)$, calculate MGF of $T$.

first hitting time of BM with drift

May directly use OST without checking the conditions.



Pf: By OST, $\mathbb{E}X_T = \mathbb{E}X_0 = 1$

$$\mathbb{E}X_T = \mathbb{E}e^{i\theta W_T + \frac{1}{2}\theta^2 T}$$

notice that $W_T = aT + b$ so

$$\mathbb{E}\, e^{(ia\theta + \frac{1}{2}\theta^2)T + ib\theta} = 1$$

$$\mathbb{E}\, e^{\underbrace{\boxed{(ia\theta+\frac{1}{2}\theta^2)}}_{\text{set as } t}T} = e^{-ib\theta}$$

So $ia\theta + \frac{1}{2}\theta^2 = t \Rightarrow \theta = -ai \pm \sqrt{2t - a^2}$

so $\mathbb{E}e^{tT} = e^{-ab \pm b\sqrt{a^2-2t}}$ is well-defined

for $a^2 - 2t \geq 0$, i.e. $t \leq \frac{1}{2}a^2$.

Since MGF takes value 1 when $t=0$, and $a<0$,

$$\mathbb{E}e^{tT} = e^{-ab - b\sqrt{a^2-2t}}$$

Rem: $M_T'(t) = e^{-ab-b\sqrt{a^2-2t}} \cdot \frac{b}{\sqrt{a^2-2t}}$

$\mathbb{E}T = M_T'(0) = \frac{b}{|a|} = -\frac{b}{a}$ (intuitive)

Another way given by Girsanov.

## 13.4.3: (Law of Arcsine)

$T$ be last hitting time to zero in time interval $[0,t]$, show $\forall u \in [0,t]$, $\mathbb{P}(T \le u) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{t}}$.

**Pf**:


$$\mathbb{P}(T \le u) = \mathbb{P}(\forall s \in (u,t],\ B_s \neq 0)$$

$$= 2 \cdot \mathbb{P}(\forall s \in (u,t],\ B_s > 0)$$

$$= 2 \cdot \mathbb{E}\left[\mathbb{P}(\forall s \in (u,t],\ B_s > 0 \mid B_u)\right]$$

Calculate $\mathbb{P}(\forall s \in (u,t],\ B_s > 0 \mid B_u = k)$

$$= \mathbb{P}\left(\forall s \in (0,t-u],\ B^u_s \geq -k\right)$$

Markov property: $B^u_t = B_{u+t} - B_u$ is BM.

$$= \mathbb{P}(\forall s \in (0,t-u],\ W_s \le k)$$

$$= \mathbb{P}(T^W_k \geq t-u)$$ the first hitting time to $k$ of $\{W_t\}$ never happens.

Clearly, $T_K^W$ has pdf $f_{T_k^W}(t) = \frac{k}{\sqrt{2\pi t^3}} e^{-\frac{k^2}{2t}}$ $(t \geq 0)$,

$$\mathbb{P}(T_K^W \geq t-u) = \int_{t-u}^{+\infty} \frac{k}{\sqrt{2\pi s^3}} e^{-\frac{k^2}{2s}} \, ds$$

So

$$\mathbb{P}(T \leq u) = 2 \cdot \int_0^{+\infty} \mathbb{P}(T_K^W \geq t-u) \cdot f_{B_u}(k) \, dk$$

$B_u \sim N(0, u)$

$$= 2 \cdot \int_0^{+\infty} \int_{t-u}^{+\infty} \frac{k}{\sqrt{2\pi s^3}} e^{-\frac{k^2}{2s}} \, ds \cdot \frac{1}{\sqrt{2\pi u}} e^{-\frac{k^2}{2u}} \, dk$$

$$\overset{\text{Fubini}}{=} \frac{1}{\pi} \int_{t-u}^{+\infty} \frac{1}{\sqrt{u}} s^{-\frac{3}{2}} \underbrace{\int_0^{+\infty} k \cdot e^{-\left(\frac{1}{2s}+\frac{1}{2u}\right)k^2} \, dk}_{\frac{1}{\frac{1}{s}+\frac{1}{u}} = \frac{us}{u+s}} \, ds$$

$(s > t-u,\ k > 0)$

$$= \frac{1}{\pi} \cdot \int_{t-u}^{+\infty} \sqrt{\frac{u}{s}} \cdot \frac{1}{u+s} \, ds$$

$$= \frac{1}{\pi} \cdot 2 \arctan \sqrt{\frac{s}{u}} \Bigg|_{s=t-u}^{+\infty}$$

$$= \frac{2}{\pi} \Big( \underbrace{\frac{\pi}{2} - \arctan \sqrt{\frac{t-u}{u}}}_{=\theta} \Big)$$

$$= \frac{2}{\pi} \cdot \arcsin \sqrt{\frac{u}{t}} \quad \checkmark$$

$$\boxed{\begin{gathered} \cot\theta \\ \| \\ \tan\left(\frac{\pi}{2}-\theta\right) = \sqrt{\frac{t-u}{u}}, \\ \sin\theta = \sqrt{\frac{u}{t}} \\ \theta = \arcsin\sqrt{\frac{u}{t}} \end{gathered}}$$

## Extra : Law of iterated logarithm

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad a.s.$$

Let $h(t) \triangleq \sqrt{2t\log\log t}$, $S_t \triangleq \sup_{s\le t}\{B_s\}$.

**Step 1**: Show $\forall t>0$, $\mathbb{P}(S_t > u\sqrt{t}) \sim \frac{2}{u\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

$(u \to +\infty)$

**Pf**: By reflection principle, $S_t \overset{d}{=} |B_t|$ so

$$\mathbb{P}(S_t > u\sqrt{t}) = 2\cdot\mathbb{P}(B_t > u\sqrt{t}) = 2\cdot\left[1 - \Phi(u)\right]$$

$$= 2\cdot\int_u^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\lim_{u\to+\infty} \frac{\mathbb{P}(S_t > u\sqrt{t})}{\frac{2}{u\sqrt{2\pi}} e^{-\frac{u^2}{2}}} = \lim_{u\to+\infty} \frac{\int_u^{+\infty} e^{-\frac{t^2}{2}} dt}{\frac{1}{u} e^{-\frac{u^2}{2}}}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{u\to+\infty} \frac{-1}{-1-\frac{1}{u^2}} = 1 \quad \checkmark$$

**Step 2**: $r$ and $c$ be any real numbers s.t. $1<r<c^2$.

Consider asymptotics of $\mathbb{P}(S_{r^n} > c\cdot h(r^{n-1}))$ and prove $\limsup_{t\to\infty} \frac{B_t}{h(t)} \le 1$ a.s.

**pf**:

$$\mathbb{P}(S_{r^n} > c\cdot h(r^{n-1})) = \mathbb{P}\left(\frac{S_{r^n}}{\sqrt{r^n}} > \sqrt{2}c\cdot\frac{1}{\sqrt{r}}\cdot\sqrt{\log(n-1)+\log\log r}\right)$$

$\frac{c^2}{r}>1$

by step 1
$$\sim \sqrt{\frac{r}{c\pi}}\cdot\frac{1}{\sqrt{\log(n-1)+\log\log r}}\cdot\underbrace{(n-1)^{-\boxed{\frac{c^2}{r}}}}_{\text{dominate order}}\cdot(\log r)^{-\frac{c^2}{r}} \quad (n\to\infty)$$

So: $\sum_{n=1}^{\infty}\mathbb{P}(S_{r^n} > c\cdot h(r^{n-1})) < \infty$, by Borel-Cantelli, $\mathbb{P}(S_{r^n} > c\cdot h(r^{n-1})\ \text{i.o.}) = 0$,

a.s. $\frac{B_{r^n}}{h(r^{n-1})} \le \frac{S_{r^n}}{h(r^{n-1})} \le c$ eventually for $\forall\, 1<r<c^2$, so set $c\to 1$ to see

$$\limsup_{t\to\infty}\frac{B_t}{h(t)} \le 1 \quad \text{a.s.}$$

**Step 3**: Show that a.s. there are infinitely many values of $n$ such that $B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r-1}{r}}\, h(r^n)$.

**Pf**: The sequence $\{B_{r^n} - B_{r^{n-1}}\}$ is independent due to independent increment of BM, so it suffices to prove by Borel-Cantelli that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r-1}{r}}\, h(r^n)\right) = \infty.$$

Estimate the order of the probability:

$$B_{r^n} - B_{r^{n-1}} \sim N(0, r^n - r^{n-1})$$

So
$$\mathbb{P}\left(B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r-1}{r}}\, h(r^n)\right) = 1 - \Phi\left(\sqrt{2}\cdot\sqrt{\log n + \log\log r}\right)$$

$$= \int_{\sqrt{2}\cdot\sqrt{\log n+\log\log r}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt \geq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{n \log r} \cdot \left[\left(\sqrt{2}\cdot\sqrt{\log n+\log\log r}\right)^{-1} - \left(\sqrt{2}\cdot\sqrt{\log n+\log\log r}\right)^{-3}\right]$$

$$\left(\int_x^{+\infty} e^{-\frac{y^2}{2}}\, dy \geq \left(\frac{1}{x} - \frac{1}{x^3}\right)\cdot e^{-\frac{x^2}{2}}\right)$$

Since $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\log n}} = +\infty$, $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{\frac{3}{2}}} < +\infty$, the result is proved. This implies by def that

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} \geq 1 \text{ a.s.} \quad \left(\text{since } \sqrt{\frac{r-1}{r}} < 1\right)$$

Combine them to see that

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad a.s.$$

$$\Big\downarrow \text{ symmetry}$$

$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = -1 \quad a.s.$$