Ito formula:

Cts semi-MG $\{X_t\}$, $f \in C^2$, then

$$df(X_t) = f'(X_t)\,dX_t + \underbrace{\tfrac{1}{2} f''(X_t)\, d\langle X, X\rangle_t}_{\text{Ito correction}}$$

in integral form: (actually well-defined)

$$X_t = \int_0^t df(X_s)$$

$$= \int_0^t f'(X_s)\,dX_s + \tfrac{1}{2}\int_0^t f''(X_s)\, d\langle X, X\rangle_s$$

$\Downarrow$

generalized to

$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f \in C^{1,2}$,

meaning $df(t, X_t) = $ ---- ——

$\Downarrow$

generalized to multi-dimensional setting where $\{X_t\}$ takes values in $\mathbb{R}^d$

eg: X, Y indep BM, $R_t^2 = X_t^2 + Y_t^2$, show that

$$Z_t = \int_0^t \frac{X_s}{R_s} dX_s + \int_0^t \frac{Y_s}{R_s} dY_s \text{ is BM.}$$

Show that $R_t^2 = 2\int_0^t R_s dZ_s + 2t$.

Pf: filtration $\mathcal{G}_t^Z = \sigma(X_s, Y_s : s \in [0,t])$,

$\{Z_t\}$ adapted to $\{\mathcal{G}_t^Z\}$ and is a cts MG.

Ito integral, check that $\forall t$,

$$\mathbb{E}\left\langle \int_0^\cdot \frac{X_s}{R_s} dX_s, \int_0^\cdot \frac{X_s}{R_s} dX_s \right\rangle_t < \infty$$

$$= \int_0^t \mathbb{E}\left(\frac{X_s}{R_s}\right)^2 ds \quad ✓$$

By Levy's characterization, suffices to check

$$\langle Z, Z\rangle_t = t.$$

$$\langle Z, Z\rangle_t = \left\langle \int_0^\cdot \frac{X_s}{R_s} dX_s, \int_0^\cdot \frac{X_s}{R_s} dX_s \right\rangle_t +$$

$$2 \cdot \underline{\left\langle \int_0^\cdot \frac{X_s}{R_s} dX_s, \int_0^\cdot \frac{Y_s}{R_s} dY_s \right\rangle_t} = 0 \text{ since } \langle X, Y\rangle_s = 0.$$

$$+ \left\langle \int_0^\cdot \frac{Y_s}{R_s} dY_s, \int_0^\cdot \frac{Y_s}{R_s} dY_s \right\rangle_t$$

$$= \int_0^t \left(\frac{X_s}{R_s}\right)^2 ds + \int_0^t \left(\frac{Y_s}{R_s}\right)^2 ds = t \quad \checkmark$$

$(X_s^2 + Y_s^2 = R_s^2)$

$R_t^2 = f(X_t, Y_t)$, $\quad f(x,y) = x^2 + y^2 \in C^2$,

Ito: $dR_t^2 = 2X_t\, dX_t + 2Y_t\, dY_t +$

$\frac{1}{2}\cdot 2 \cdot d\langle X, X\rangle_t + \frac{1}{2}\cdot 2\cdot 0 \cdot d\langle X, Y\rangle_t + \frac{1}{2}\cdot 2 \cdot d\langle Y, Y\rangle_t$

$= 2X_t\, dX_t + 2Y_t\, dY_t + 2dt$

Integrath both sides:

$$R_t^2 - R_0^2 = 2\cdot\left(\int_0^t X_s\, dX_s + \int_0^t Y_s\, dY_s\right) + 2t$$

$$= 2\cdot\left(\int_0^t R_s\, dZ_s\right) + 2t$$

$(R_t\, dZ_t = X_t\, dX_t + Y_t\, dY_t$ proved$)$

Remark: For SDE $dR_t^2 = 2R_t\, dW_t + 2dt$, there exists a weak solution under $W_t = Z_t$, $\mathcal{F}_t = \mathcal{F}_t^Z$ given by the construction above. However, it's not guaranteed that a strong solution (pathwise) exists!

e.g.: $\begin{cases} dX_t = dW_t \\ X_0 = 0 \end{cases}$ Strong solution : $X_t = W_t$

Weak solution : $\begin{cases} (\tilde{X}_t = B_t,\ \tilde{W}_t = B_t,\ \tilde{\mathcal{F}}_t = \sigma(B_s : s\in[0,t])) \\ (\tilde{X}_t = -B_t,\ \tilde{W}_t = -B_t,\ \tilde{\mathcal{F}}_t = \sigma(B_s : s\in[0,t])) \end{cases}$

$\downarrow$ can change probability space, BM, and filtration.

$\Downarrow$ $\mathbb{P}(B_t = -B_t) = 0$

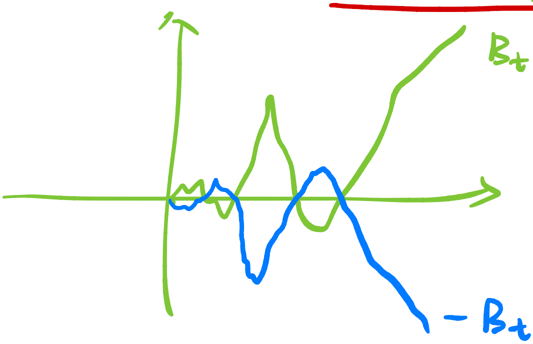
- Strong solution is unique
- Weak solution not unique, even zero probability of taking the same value at a fixed time!
- However, weak solution unique under the sense of finite-dim dist since

$$\{B_t\} \overset{d}{=} \{-B_t\}$$

same finite-dim dist, both BM

$\not\Downarrow$

same pathwise behavior



Another example showing problems caused by measurability:

e.g: $\{W_t\}$ BM, then time-reversal $\{t \cdot W_{\frac{1}{t}}\}$ is BM.

Easily proved by viewing as a GP.

However, what's the filtration of $Y_t = t \cdot W_{\frac{1}{t}}$, it's $\underbrace{\mathcal{F}_t^Y = \sigma(W_s : s \in [\tfrac{1}{t}, +\infty))}_{\text{cost of time reversal.}}$

Obviously, $\{Y_t\}$ is MG under $\{\mathcal{F}_t^Y\}$ but not MG under $\{\mathcal{F}_t^W\}$ since

$$\mathbb{E}(Y_2 \mid \mathcal{F}_1^W) = 2 \cdot \mathbb{E}(W_{\frac{1}{2}} \mid \mathcal{F}_1^W)$$

$$= 2 W_{\frac{1}{2}} \neq W_1 = Y_1 .$$

$$\Downarrow$$

a way to prove $\{\mathcal{F}_t^W\} \neq \{\mathcal{F}_t^Y\}$

you may see this in the qual!

e.g: Find SDE of the following process:

(a): $X_t = \frac{1}{1+t} W_t = f(t, W_t)$

where $f(t,x) = \frac{1}{1+t} x$

$$dX_t = -\frac{X_t}{(1+t)^2} dt + \frac{1}{1+t} dW_t + \frac{1}{2} \cdot 0 \cdot d\langle W, W\rangle_t$$

(b): $X_t = \sin W_t = f(W_t)$

where $f(x) = \sin x$

$$dX_t = \cos W_t \, dW_t + \frac{1}{2} \cdot (-\sin W_t) \, d\langle W, W\rangle_t$$

$$= -\frac{1}{2} \sin W_t \, dt + \cos W_t \, dW_t$$

$$= -\frac{1}{2} X_t \, dt + \cos W_t \, dW_t$$

(c): $\begin{cases} X_t = a \cos W_t \\ Y_t = b \sin W_t \end{cases}$ $\quad \frac{X_t^2}{a^2} + \frac{Y_t^2}{b^2} = 1$ (ellipse) $\quad (ab \neq 0)$

$$\begin{cases} dX_t = -a \sin W_t \, dW_t + \frac{1}{2}(-a \cos W_t) \, dt \\ dY_t = b \cos W_t \, dW_t + \frac{1}{2}(-b \sin W_t) \, dt \end{cases}$$

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} 0 & -\frac{a}{b} \\ \frac{b}{a} & 0 \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dW_t$$

Solve SDE:

e.g.: $dX_t = \underbrace{-P_t X_t \, dt}_{\text{OU dynamics}} + \sigma \, dW_t$ where X_t takes value in $\mathbb{R}^n$, P_t is deterministic and takes value in $\mathbb{R}^{n\times n}$, W is BM on $\mathbb{R}^n$.

Assumption: $\underbrace{\forall s, t \in [0,T],\ P_t \text{ commute with } P_s.}_{\text{necessary!}}$

Pf:

first solve ODE $dX_t = -P_t X_t \, dt$

here we need commutative assumption to get

$$X_t = C \cdot \underbrace{e^{-\int_0^t P_s ds}}_{\substack{\text{o.w. this matrix} \\ \text{exponential is wrong!}}}$$

Then change C from constant to a process $\{C_t\}$,

$$X_t = C_t \cdot e^{-\int_0^t P_s ds},$$

Ito: $$dX_t \overset{\text{product rule}}{=} e^{-\int_0^t P_s ds} \, dC_t + C_t \cdot d\, e^{-\int_0^t P_s ds}$$

$$\underbrace{+\, d\langle C, \underbrace{e^{-\int_0^{\cdot} P_s ds}}_{\text{finite variation}} \rangle_t}_{=0}$$

$$= e^{-\int_0^t P_s\,ds}\,dC_t - P_t \cdot \underbrace{C_t \cdot e^{-\int_0^t P_s\,ds}}_{X_t}\,dt$$

$$\overset{\text{original}}{\underset{\text{SDE}}{=}} -P_t X_t\,dt + \sigma\,dW_t$$

Get: $$dC_t = e^{\int_0^t P_s\,ds} \cdot \sigma\,dW_t$$

$$C_t = \underbrace{C}_{\substack{\downarrow \\ \text{some deterministic} \\ \text{constant } C_0}} + \int_0^t e^{\int_0^s P_u\,du} \cdot \sigma \cdot dW_s$$

So: $$X_t = C \cdot e^{-\int_0^t P_u\,du} + \sigma \cdot \int_0^t e^{-\int_s^t P_u\,du}\,dW_s$$

match initial condition X_0

$$\Downarrow$$

$$X_t = X_0 \cdot e^{-\int_0^t P_u\,du} + \sigma \cdot \int_0^t e^{-\int_s^t P_u\,du}\,dW_s$$

dist: $$X_t \sim N\left(X_0 \cdot e^{-\int_0^t P_u\,du},\ \sigma^2 \cdot \int_0^t e^{-2\int_s^t P_u\,du}\,ds\right)$$