12.1.5: $(Y, \mathcal{G})$ be $\mathcal{L}^2$-MG, show that $\forall\, i \le j \le k$, $\mathbb{E}[(Y_k - Y_j)Y_i] = 0$ and

$$\mathbb{E}\left((Y_k - Y_j)^2 \mid \mathcal{G}_i\right) = \mathbb{E}(Y_k^2 \mid \mathcal{G}_i) - \mathbb{E}(Y_j^2 \mid \mathcal{G}_i).$$

If $\exists K$, $\forall n$, $\mathbb{E}Y_n^2 \le K$, show that (Y_n) converge in $\mathcal{L}^2$.

Pf:

$$\mathbb{E}[(Y_k - Y_j)Y_i] = \mathbb{E}\Big[Y_i \underbrace{\mathbb{E}(Y_k - Y_j \mid \mathcal{G}_i)}_{\substack{= Y_i - Y_i = 0 \\ (\text{MG})}}\Big] = 0$$

$$\mathbb{E}\left((Y_k - Y_j)^2 \mid \mathcal{G}_i\right) = \mathbb{E}(Y_k^2 \mid \mathcal{G}_i) - 2\underbrace{\mathbb{E}(Y_k Y_j \mid \mathcal{G}_i)} + \mathbb{E}(Y_j^2 \mid \mathcal{G}_i)$$

$$\mathbb{E}(Y_k Y_j \mid \mathcal{G}_i) = \mathbb{E}(Y_j^2 \mid \mathcal{G}_i) + \mathbb{E}[Y_j(Y_k - Y_j) \mid \mathcal{G}_i]$$

$$= \mathbb{E}(Y_j^2 \mid \mathcal{G}_i) + \mathbb{E}\left(\mathbb{E}[Y_j(Y_k - Y_j) \mid \mathcal{G}_j] \mid \mathcal{G}_i\right)$$

$$= \mathbb{E}(Y_j^2 \mid \mathcal{G}_i) + \mathbb{E}\Big(Y_j \cdot \underbrace{\mathbb{E}(Y_k - Y_j \mid \mathcal{G}_j)}_{=0} \mid \mathcal{G}_i\Big)$$

$$= \mathbb{E}(Y_j^2 \mid \mathcal{G}_i)$$

$$= \mathbb{E}(Y_k^2 \mid \mathcal{G}_i) - \mathbb{E}(Y_j^2 \mid \mathcal{G}_i)$$

If $\exists K, \forall n, \mathbb{E} Y_n^2 \le K$, then $\sup_n \mathbb{E} Y_n^2 \le K < \infty$, $\{Y_n\}$ converge in $\mathcal{L}^2$ by MG $\mathcal{L}^2$-convergence thm.

<u>12.1.7</u>: $\{Y_n\}$ sub-MG, u convex $\uparrow$ from $\mathbb{R}$ to $\mathbb{R}$, show that $\{u(Y_n)\}$ is sub-MG if $\mathbb{E}|u(Y_n)| < \infty$ for $\forall n$. Show that Y_n^+ is sub-MG but $|Y_n|$ is not necessarily sub-MG.

<u>Pf</u>: $\{Y_n\}$ equipped with filtration $\{\mathcal{F}_n\}$.

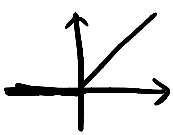
Check: ①: $\{u(Y_n)\}$ adapted to $\{\mathcal{F}_n\}$ ✓

②: $\forall n$, $\mathbb{E}|u(Y_n)| < \infty$ ✓

③: $\mathbb{E}[u(Y_{n+1}) | \mathcal{F}_n] \underset{\text{Jensen}}{\geq} u(\mathbb{E}(Y_{n+1} | \mathcal{F}_n))$

$\underset{u \uparrow,\ \text{sub-MG}}{\geq} u(Y_n)$

so $\{u(Y_n)\}$ is sub-MG.

$u(x) = \max\{x, 0\}$  convex, $\uparrow$ so $\{Y_n^+\}$ is sub-MG

Counter e.g.: $\{S_n\}$ SRW, up w.p. p, down w.p. q,

$\mathbb{E}(|S_{n+1}| | \mathcal{F}_n) = p \cdot |S_n + 1| + q \cdot |S_n - 1|$

Now consider the case $p > \frac{1}{2}$, $S_n \le -1$, then

$$\mathbb{E}(|S_{n+1}| \mid \mathcal{F}_n) = p \cdot (|S_n| - 1) + q \cdot (|S_n| + 1)$$

$$= |S_n| + \underbrace{q - p}_{<0} < |S_n| .$$

<u>**12.1.8**</u>: $\{X_n\}$ Markov chain, countable state space S with transition matrix P. If $\psi : S \to \mathbb{R}$ bounded and $\sum_{j \in S} P_{ij} \psi(j) \le \lambda \cdot \psi(i)$ for some $\lambda > 0$ and $\forall i \in S$. Show that $\lambda^{-n} \cdot \psi(X_n)$ is a super-MG.

<u>Pf</u>: Check: ①: $\{\lambda^{-n} \cdot \psi(X_n)\}$ adapted to $\{\mathcal{F}_n\}$

$\{X_n\}$ adapted to this

②: $\forall n$, $\mathbb{E}|\lambda^{-n} \cdot \psi(X_n)| = \lambda^{-n} \cdot \underbrace{\mathbb{E}|\psi(X_n)|}_{<\infty}$

since ψ is bounded

③: $\mathbb{E}(\lambda^{-n-1} \cdot \psi(X_{n+1}) \mid \mathcal{F}_n) = \lambda^{-n-1} \cdot \mathbb{E}(\psi(X_{n+1}) \underbrace{\mid X_n}_{\text{Markov}})$

$= \lambda^{-n-1} \cdot \underline{\sum_{j \in S} \psi(j) \cdot P_{X_n, j}} \le \lambda^{-n-1} \cdot \lambda \cdot \underline{\psi(X_n)}$

$= \lambda^{-n} \cdot \psi(X_n)$ proves it's a super-MG.

Super-MG $\xleftrightarrow[\text{chain}]{\text{Markov}}$ super-harmonic func ($\Delta \le 0$)

12.1.9 : $G_n(s)$ is prob g.f. of Z_n (branching process) where $Z_0=1$, $Var(Z_1)>0$. Let H_n be inverse of G_n as a func defined on $[0,1]$, show that $M_n=(H_n(s))^{Z_n}$ defines a MG w.r.t. the filtration generated by $\{Z_n\}$.

Pf : check : ① adapted ✓

② : $\forall n$, $\mathbb{E}|M_n| = \mathbb{E}(\underbrace{H_n(s)}_{\in[0,1]})^{Z_n} < \infty$

③ : $\mathbb{E}(M_{n+1}|\mathcal{G}_n) = \mathbb{E}([H_{n+1}(s)]^{Z_{n+1}}|\mathcal{G}_n)$

$$= \mathbb{E}\left[[H_{n+1}(s)]^{Z_{n+1}} \Big| Z_n\right]$$

$\{Z_n\}$ is Markov

$$Z_{n+1} = \underbrace{\xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1}}_{\text{i.i.d. follow offspring dist, indep of } Z_n}$$

$$= \mathbb{E}\left([H_{n+1}(s)]^{\xi_1^{n+1}} \cdot \cdots \cdot [H_{n+1}(s)]^{\xi_{Z_n}^{n+1}} \Big| Z_n\right)$$

Consider

$$\mathbb{E}\left([H_{n+1}(s)]^{\xi_1^{n+1}} \cdot \cdots \cdot [H_{n+1}(s)]^{\xi_k^{n+1}} \Big| Z_n = k\right)$$

$$= \mathbb{E}\left([H_{n+1}(s)]^{\xi_1^{n+1}} \cdot \cdots \cdot [H_{n+1}(s)]^{\xi_k^{n+1}}\right)$$

$$= \left(\mathbb{E}\,[H_{n+1}(s)]^{\xi_1^{n+1}}\right)^k = \left[\sum_{j=0}^{\infty}[H_{n+1}(s)]^j \cdot \mathbb{P}(\xi_1^{n+1}=j)\right]^k$$

$$= \left[\underline{G_1(H_{n+1}(s))} \right]^k$$

by def, $G_1(s) = \mathbb{E} s^{Z_1}$

$$= \sum_{j=0}^{\infty} s^j \cdot \underbrace{\mathbb{P}(Z_1 = j)}_{\text{offspring dist}}$$

since $H_{n+1} = (G_{n+1})^{-1}$ and $G_{n+1} = \underbrace{G_1 \circ G_1 \circ \cdots \circ G_1}_{(n+1) \text{ of } G_1}$

so $H_{n+1} = (G_n \circ G_1)^{-1} = G_1^{-1} \circ G_n^{-1} = G_1^{-1} \circ H_n$

now $= [H_n(s)]^k$ so

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = [H_n(s)]^{Z_n} = M_n \quad ✓$$

☆: MG structure is hard to find typically, but if found, it tells a lot!

Try this question: $\{Z_n\}$ be branching process, $Z_0 = 1$, suppose offspring mean $\mu < 1$, $\mathbb{P}(Z_1 \geq 2) > 0$, show that $\mathbb{E}(\sup_n Z_n) \leq \frac{y}{y-1}$ where y is the largest root of $x = G(x)$, G is g.f. of offspring dist

Hint: ①: check $y > 1$ ②: $Y_n = y^{Z_n}$ is positive MG

③: Doob's maximal inequality ④: Tail formula for expectation

Hoeffding & Azuma's inequality

Chernoff bound
(Markov + optimize bound)

Hoeffding's lemma
(a.s. bounded r.v. is sub-Gaussian)

Hoeffding's inequality
(independent r.v.)

Azuma's inequality
(MG increments)

Thm (Azuma): $(Y, \mathcal{G})$ is MG, if $\exists\, K_1, K_2, \dots$ such that $\underbrace{|Y_n - Y_{n-1}| \leq K_n}_{\text{a.s. bounded MG increments}}$ a.s. for $\forall n$, then

$$\forall x > 0,\ \mathbb{P}\big(\underbrace{|Y_n - Y_0|}_{\text{sum of MG increments}} \geq x\big) \leq 2 \cdot e^{-\frac{x^2}{2 \cdot \sum_{i=1}^{n} K_i^2}}$$

12.2.1: (Knapsack)

n objects with volume V_i, worth W_i, (all indep r.v.!)
$\exists M, \forall i, W_i \le M$ a.s. Knapsack has volume c,
maximize the worth packed in it within volume c.

Find $z_1, \dots, z_n \in \{0,1\}$, $\begin{cases} \max \sum_i z_i W_i \\ \text{s.t. } \sum_i z_i V_i \le c \end{cases}$

Let Z be maximal possible worth in knapsack, prove

$$\mathbb{P}(|Z - \mathbb{E}Z| \ge x) \le 2e^{-\frac{x^2}{2nM^2}} \quad (\forall x > 0)$$

Pf: $\mathcal{F}_i = \sigma(V_1, \dots, V_i, W_1, \dots, W_i)$, set
$Y_i = \mathbb{E}(Z|\mathcal{F}_i)$ as a MG, wish to prove
$|Y_j - Y_{j-1}| = |\mathbb{E}(Z|\mathcal{F}_j) - \mathbb{E}(Z|\mathcal{F}_{j-1})|$ is a.s. bounded

Consider X_j as the maximal worth in knapsack
without j-th item, then $\mathbb{E}(X_j|\mathcal{F}_j) = \mathbb{E}(X_j|\mathcal{F}_{j-1})$,
connects $\mathcal{F}_j$ and $\mathcal{F}_{j-1}$

since $X_j \le Z \le X_j + M$, we know

$$\begin{cases} \mathbb{E}(X_j|\mathcal{F}_j) \le Y_j \le \mathbb{E}(X_j|\mathcal{F}_j) + M \\ \mathbb{E}(X_j|\mathcal{F}_{j-1}) \le Y_{j-1} \le \mathbb{E}(X_j|\mathcal{F}_{j-1}) + M \end{cases} \Rightarrow |Y_j - Y_{j-1}| \le M$$

use Azuma, concludes the proof.

12.2.2 (Graph coloring)

n-vertex Erdos-Rényi graph (each edge prob p appear)
The chromatic number χ is the minimal num of color needed such that each vertex colored differently from its neighbors. Show $\mathbb{P}(|\chi - \mathbb{E}\chi| \geq x) \leq 2 \cdot e^{-\frac{x^2}{2n}}$ $(\forall x > 0)$.

Pf:

$$\mathcal{G}_i = \sigma\big((v_a, v_b) : 1 \leq a, b \leq i\big), \quad Y_i = \mathbb{E}(\chi \mid \mathcal{G}_i) \quad \text{MG}$$

wish to prove $|Y_j - Y_{j-1}|$ is a.s. bounded.

Consider X_j as the minimal # of color needed in the graph with v_j deleted, then

$$X_j \leq \chi \leq X_j + 1, \quad \mathbb{E}(X_j \mid \mathcal{G}_j) = \mathbb{E}(X_j \mid \mathcal{G}_{j-1})$$

now
$$\begin{cases} \mathbb{E}(X_j \mid \mathcal{G}_j) \leq Y_j \leq \mathbb{E}(X_j \mid \mathcal{G}_j) + 1 \\ \mathbb{E}(X_j \mid \mathcal{G}_{j-1}) \leq Y_{j-1} \leq \mathbb{E}(X_j \mid \mathcal{G}_{j-1}) + 1 \end{cases}$$

so $|Y_j - Y_{j-1}| \leq 1$, Azuma concludes the proof.