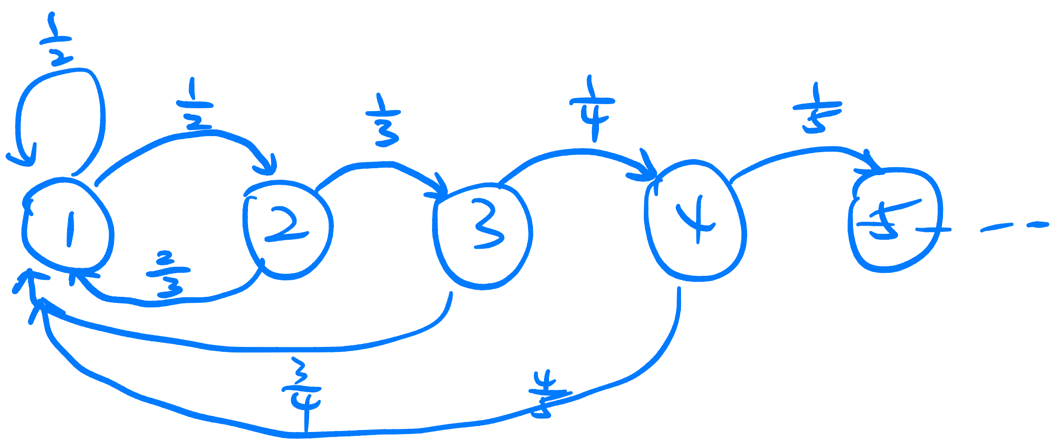
eg: $\{X_n\}$ MC, $\underbrace{S=\{1,2,3,4,\cdots\}}_{\text{infinite state space}}$,

$$P=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & - & - & - \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & - & - & - \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & - \\ \frac{4}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & \cdots \\ \vdots & \ddots & & & \ddots & & \\ & & \ddots & & & \ddots & \end{pmatrix}$$

↓ rigorously, this is not a matrix since its dimension is ∞.

write
$$P_{ij}=\begin{cases} \frac{i}{i+1} & \text{if } j=1 \\ \frac{1}{i+1} & \text{if } j=i+1 \\ 0 & \text{else} \end{cases}$$



(a): Determine if it's irreducible.

any pair of states communicate (only one communication class)

$\Downarrow$

irreducible ✓

(b): classify all states (transient / recurrent)

Main tool: if a communication class is closed and it contains finitely many states, then it's recurrent.

$\Uparrow \times$

(problem: S infinite set)

now this Markov chain is irreducible and closed, but this does not necessarily imply recurrence.

recall: state s is recurrent if

$$\mathbb{P}_{X_0=s}(T_s<\infty)=1.$$

Irreducible $\Longrightarrow$ either all states are recurrent
or all states are transient.

So: only need to focus on the recurrence
property of one state.

$\Downarrow$

investigate recurrence property
of state 1
only focus

Define $T_1 \triangleq \inf\{n \geq 1 : X_n = 1\}$
(first hitting time back to state 1)
(except time 0)

want to find $\mathbb{P}_{X_0=1}(T_1 < \infty)$.

$$\mathbb{P}_{X_0=1}(T_1 = 1) = \frac{1}{2}$$

$$\mathbb{P}_{X_0=1}(T_1 = 2) = \mathbb{P}_{X_0=1}(X_1 \neq 1, X_2 = 1) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$

$$\mathbb{P}_{X_0=1}(T_1=k) = \mathbb{P}_{X_0=1}(\underbrace{X_1\neq 1}_{\Updownarrow\, X_1=2}, X_2\neq 1, \dots, X_{k-1}\neq 1, X_k=1)$$

$(\forall k\geq 1,\ k\in\mathbb{Z})$

$$= \mathbb{P}_{X_0=1}(X_1=2, \underbrace{X_2\neq 1}_{\Updownarrow\, X_2=3}, \dots, X_{k-1}\neq 1, X_k=1)$$

since under the event $\{X_1=2\}$, X_2 can be either 1 or 3

$$= \mathbb{P}_{X_0=1}(X_1=2, X_2=3, X_3\neq 1, \dots, X_{k-1}\neq 1, X_k=1)$$

(apply same trick)

$$= \mathbb{P}_{X_0=1}(X_1=2, X_2=3, X_3=4, \dots, X_{k-1}=k, X_k=1)$$

decompose into product of transition prob

$$= P_{12}\cdot P_{23}\cdot P_{34}\cdot \dots \cdot P_{k-1,k}\cdot P_{k1}$$

$$= \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\dots\times\frac{1}{k}\times\frac{k}{k+1}$$

$$= \frac{k}{(k+1)!}$$

PMF of T_1.

$$\mathbb{P}_{X_0=1}(T_1<\infty)$$ (be careful that stopping time can take value ∞)

$$= \sum_{k=1}^{\infty} \mathbb{P}_{X_0=1}(T_1=k)$$

$$= \sum_{k=1}^{\infty} \frac{k}{(k+1)!}$$ (Two ways to verify that $\mathbb{P}_{X_0=1}(T_1<\infty)=1$)

Way 1: Calculate series.

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = \sum_{k=1}^{\infty} \frac{(k+1)-1}{(k+1)!}$$

$$= \sum_{k=1}^{\infty} \left[\frac{1}{k!} - \frac{1}{(k+1)!}\right]$$

$$= \frac{1}{1!} \cancel{- \frac{1}{2!}} \cancel{+ \frac{1}{2!}} \cancel{- \frac{1}{3!}} \cancel{+ \cdots}$$

$$= 1$$

So: state 1 is recurrent,
the whole MC is recurrent.

Way 2:

To consider $\mathbb{E}_{X_0=1} T_1$.

$$\mathbb{E}_{X_0=1} T_1 = \sum_{k=1}^{\infty} k \cdot \mathbb{P}_{X_0=1}(T_1 = k)$$

$$= \sum_{k=1}^{\infty} k \cdot \frac{k}{(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{k^2}{(k+1)!} < \infty$$

only need to judge convergence, don't need to compute its limit

$\sum_{n=1}^{\infty} a_n \ (a_n > 0)$: if $\frac{a_{n+1}}{a_n} \to r \ (n \to \infty)$ and $r < 1$, then $\sum_{n=1}^{\infty} a_n < \infty$

because
$$\frac{\frac{(k+1)^2}{\cancel{(k+2)!}\,(k+2)}}{\frac{k^2}{\cancel{(k+1)!}}} = \frac{(k+1)^2}{k^2(k+2)} \to 0 < 1 \quad (k \to \infty)$$

implies that $\sum_{k=1}^{\infty} \frac{k^2}{(k+1)!}$ converges.

$$\mathbb{E}_{X_0=1} T_1 < \infty \Rightarrow \mathbb{P}_{X_0=1}(T_1 < \infty) = 1.$$

(otherwise expectation is ∞)

$\Rightarrow$ state 1 recurrent (whole MC). ✓

e.g.: $\{Y_n\}$ MC, $\{X_n\}$ MC, X_n has transition matrix P_X, $\forall k \geq 0$, $Y_k = X_{2k}$, what's the transition matrix of $\{Y_n\}$? If $\{X_n\}$ has stationary distribution π_X, what is the stat dist of $\{Y_n\}$?

Pf:

$$(P_Y)_{ij} = \mathbb{P}(Y_{k+1} = j \mid Y_k = i)$$

$$= \mathbb{P}(X_{2k+2} = j \mid X_{2k} = i)$$

$$= [(P_X)^2]_{ij}$$

two-step transition prob.

so: $P_Y = (P_X)^2$.

Assume $\{Y_n\}$ has stat dist π_Y.

$$\begin{cases} \pi_X^T \cdot P_X = \pi_X^T \\ \pi_Y^T \cdot P_Y = \pi_Y^T \end{cases}, \quad \text{know} \quad P_Y = (P_X)^2$$

actually $\pi_X = \pi_Y$.

$$\pi_X^T \cdot P_Y = \pi_X^T \cdot (P_X)^2 = \boxed{\pi_X^T \cdot P_X} \cdot P_X$$

$$= \pi_X^T \cdot P_X = \pi_X^T$$

proving $\pi_X = \pi_Y$.

Interpretation: transit 1 step under $\{Y_n\}$ is equivalent to transiting 2 steps under $\{X_n\}$. So stationary dist is the same (since the law of transition is the same, and only the rate of transition is different).