Markov chain $\Rightarrow$ state space $\mathcal{S}$

each state $s \in \mathcal{S}$ $\begin{cases} \text{recurrent state} \\ \text{transient state} \end{cases}$

State classification

Recurrence:

Def: state s is called recurrent if

$$\mathbb{P}_{X_0=s}(T_s < \infty) = 1$$

start the MC at state s

$T_s \triangleq \inf\{n \geq 1 : X_n = s\}$ is the first time that MC hits state s (except time 0)

state s is recurrent iff starting from state s, with probability 1, we can come back to state s for infinitely many times.

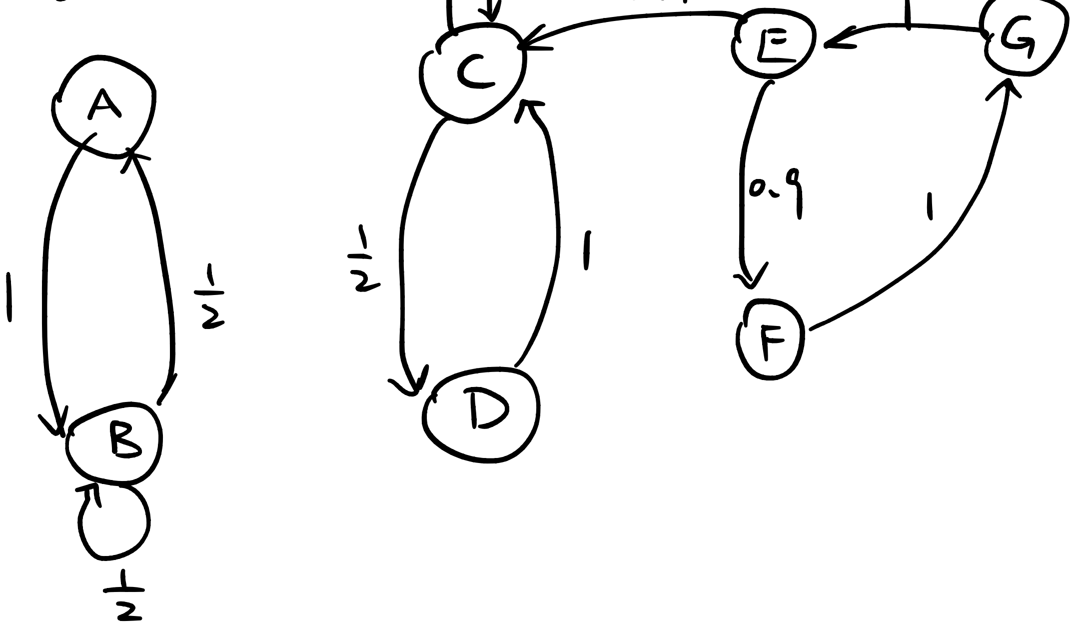
Intuitively, Markov property means that the process is memoryless, so we can restart the process at any time.

If state s is recurrent, starting from $X_0 = s$, returning back to s cost finitely many time. By Markov property, we act as if we restart the process after it hits s, so it only takes finitely many time until the second hit to s. Do this for infinitely many times, as time $\to \infty$, each hit to state s cost finite time

$\Downarrow$

infinitely many visits to state s.

e.g: $\{X_n\}$ MC,



(a): Write transition matrix

state space $S = \{A, B, C, D, E, F, G\}$

$$
\begin{array}{c|ccccccc}
 & A & B & C & D & E & F & G \\
\hline
A & & 1 & & & & & \\
B & \frac{1}{2} & \frac{1}{2} & & & & & \\
C & & & \frac{1}{2} & \frac{1}{2} & & & \\
D & & & 1 & & & & \\
E & & & 0.1 & & & 0.9 & \\
F & & & & & & & 1 \\
G & & & & & 1 & &
\end{array}
$$

(b): Is there a limiting distribution?

$$\lim_{n\to\infty} \underbrace{\alpha^T P^n}_{(X_n)}$$

Since not all states are communicating with each other, there's no single limiting dist, i.e. limiting dist depends on the initial dist α^T.

If we restrict to MC with state $\{A, B\}$ or the MC with state $\{C, D, E, F, G\}$, there is a single limiting distribution.

(c): Determine the set of stationary dist if exist.

$$\pi^T \cdot P = \pi^T$$

$$(\pi_1, ---, \pi_7) \cdot P = (\pi_1, ---, \pi_7)$$

$$\begin{cases} \frac{1}{2}\pi_2 = \pi_1 \\ \pi_1 + \frac{1}{2}\pi_2 = \pi_2 \quad (\times \text{ redundant}) \\ \frac{1}{2}\pi_3 + \pi_4 + \cancel{0.1\pi_5} = \pi_3 \quad (\times \text{ redundant}) \\ \quad (\pi_5 = 0) \\ \frac{1}{2}\pi_3 = \pi_4 \\ \pi_7 = \pi_5 \\ 0.9\pi_5 = \pi_6 \\ \pi_6 = \pi_7 \end{cases}$$

$$\pi_5 = \pi_7 = \pi_6 = 0.9\pi_5$$

$$\Downarrow$$

$$\pi_5 = 0, \ \pi_6 = \pi_7 = 0.$$

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_2 \\ \pi_4 = \frac{1}{2}\pi_3 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$\pi_4 = a$ as free variable, $\pi_3 = 2a$, $\begin{cases} \pi_1 = \frac{1}{2}\pi_2 \\ \pi_1 + \pi_2 = 1 - 3a \end{cases}$

$\frac{3}{2}\pi_2 = 1 - 3a$, $\pi_2 = \frac{2(1-3a)}{3}$, $\pi_1 = \frac{1-3a}{3}$

So: if π is stat dist,

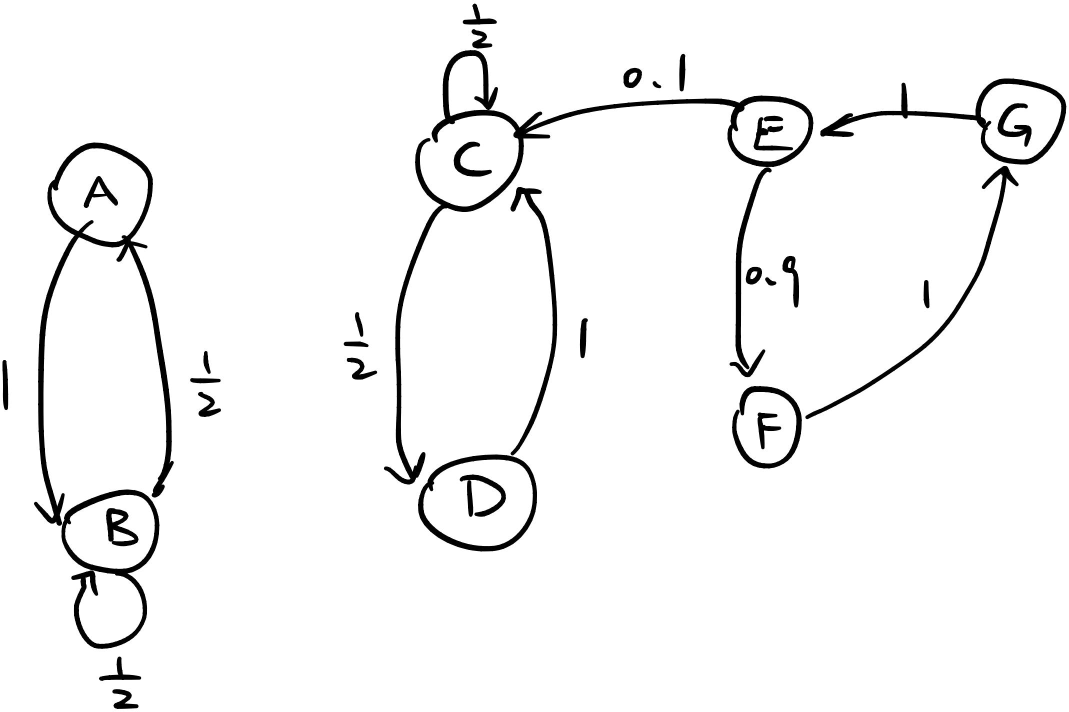
$$\pi = \begin{bmatrix} \frac{1}{3}(1-3a) \\ \frac{2}{3}(1-3a) \\ 2a \\ a \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{matrix} a \geq 0, \\ 1-3a \geq 0 \end{matrix} \right) \Downarrow \left(0 \leq a \leq \frac{1}{3} \right)$$

here stat dist is not unique, since transition graph has two connected components.

these states are transient

If s transient, it must has value 0 in stationary dist.



(d): communication class

$\{A, B\}$, $\{C, D\}$, $\{E, F, G\}$

(e): recurrent / transient states

$\{A, B\}$ is closed $\Rightarrow$ A recurrent, B recurrent

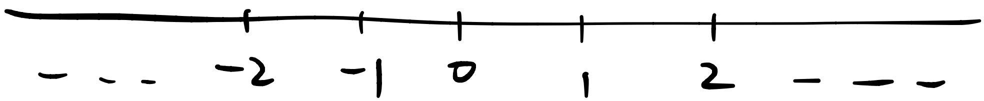
closed communication class
with finitely many states is recurrent!
(all states in the same comm class
has the same recurrence property)

$\{C, D\}$ is closed $\Rightarrow$ C, D recurrent.

$\{E, F, G\}$ is not closed $\Rightarrow$ E, F, G transient.

SRW $\Rightarrow$ MC

symmetric SRW $S = \mathbb{Z}$



since symmetric, there's no trend of going left/right for MC, $\boxed{\text{recurrent}}$.

asymmetric SRW p: prob of going right
($p \neq \frac{1}{2}$)

since asymmetric, there's a clear trend that it's either going left or going right, so starting from 0, we cannot expect to return ~~to~~ 0 again in finite time with prob 1

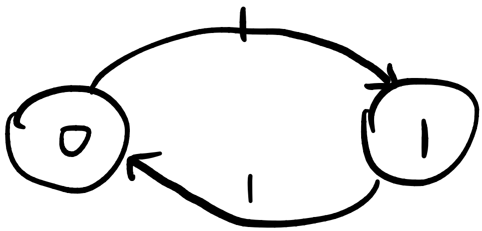
so it's $\boxed{\text{transient.}}$

$\{X_n\}$: MC, irreducible, closed, finitely many states

$$\Downarrow$$

$$\boxed{\text{recurrent}}$$

$\{X_n\}$:



if we start the chain at 0,

$X_0=0,\quad X_1=1,\quad \underbrace{X_2=0},\quad X_3=1$

$\Uparrow$

first hitting time back to 0,

so $T_0=2$.

$$\mathbb{P}_{X_0=0}(T_0<\infty)=1$$

by def, $\{X_n\}$ is $\boxed{\text{recurrent}}$.