$\{X_n\}$ MC, initial dist is α^T (dist of X_0), transition matrix P

$$\Downarrow$$

dist of X_n: $\alpha^T \cdot P^n$

$$\Downarrow$$

<u>limiting dist</u> is $\lim_{n\to\infty} \alpha^T \cdot P^n = \alpha^T \cdot \boxed{\lim_{n\to\infty} P^n}$

depends on the asymptotic behavior of P^n.

(π is always a column vector)

<u>stationary dist</u>: π^T is a stationary dist if

① : $\pi_1 + \pi_2 + \cdots + \pi_n = 1$, $\forall i$, $\pi_i \geq 0$

π gives a distribution

② : $\pi^T P = \pi^T$

if $X_n \sim \pi$, then $X_{n+1} \sim \pi$

$$\begin{tikzcd} \text{limiting dist} \arrow[d, bend right, "\text{under some conditions}"'] \\ \text{stationary dist} \arrow[u, bend right, "\text{not necessarily}"'] \end{tikzcd}$$

$$\Uparrow$$

refer to jupyter notebook for week 5, I have some examples to show the difference between those concepts.

e.g: $\{X_n\}$ MC, state space $S=\{1,2,3\}$,

$$P=\begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$, compute stationary dist π.

Pf:

$$\pi^T \cdot P = \pi^T$$

① : Plug in P, solve linear system for π.

② : Interpretation of eigenvector

①:

$$(\pi_1, \pi_2, \pi_3) \cdot \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$\begin{cases} \frac{1}{4}\pi_1 + \frac{2}{3}\pi_3 = \pi_1 \\ \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 = \pi_2 \\ \frac{3}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$\Rightarrow$ linear system for π_1, π_2, π_3

normalization

$\Downarrow$

solve it to get

$$\pi = \begin{pmatrix} \frac{8}{23} \\ \frac{6}{23} \\ \frac{9}{23} \end{pmatrix}$$

② : $\pi^T P = \pi^T$

$\downarrow$ take transpose

$(\pi^T P)^T = P^T \pi = \pi$

π is the eigenvector of P^T

correspondent to eigenvalue 1

(e.g: $Ax = \lambda x$, $x \neq 0$)

$\downarrow$

$(P^T - I) \cdot \pi = 0$

$\downarrow$

$$P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}, \quad P^T = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^T - I = \begin{pmatrix} -\frac{3}{4} & 0 & \frac{2}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{2} & -1 \end{pmatrix}$$

π is in the null space of $P^T - I$.

$$\begin{pmatrix} -\frac{3}{4} & 0 & \frac{2}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{2} & -1 \end{pmatrix} \xRightarrow[\text{row 1 add row 3}]{\text{row 2} \times(-2)} \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ \frac{3}{4} & \frac{1}{2} & -1 \end{pmatrix}$$

$$\Downarrow \text{row } 3 \times \tfrac{4}{3},\ \text{row 1} - (\text{row 2} \times \tfrac{1}{2})$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 1 & \frac{2}{3} & -\frac{4}{3} \end{pmatrix}$$

$$\begin{cases} \pi_2 - \frac{2}{3}\pi_3 = 0 \\ \pi_1 + \frac{2}{3}\pi_2 - \frac{4}{3}\pi_3 = 0 \end{cases}$$

$$\Downarrow$$

$$\boxed{\text{null space is } \begin{pmatrix} \frac{8}{9}a \\ \frac{2}{3}a \\ a \end{pmatrix}}$$

Normalization step

$$\Downarrow$$

$$\frac{8}{9}a + \frac{2}{3}a + a = 1 \quad (\pi \text{ prob dist})$$

$$\Rightarrow a = \frac{9}{23} \Rightarrow \pi = \begin{pmatrix} \frac{8}{23} \\ \frac{6}{23} \\ \frac{9}{23} \end{pmatrix}$$

same answer.

e.g.: $\{X_n\}$ MC, state space $S=\{1,2,3,4\}$,

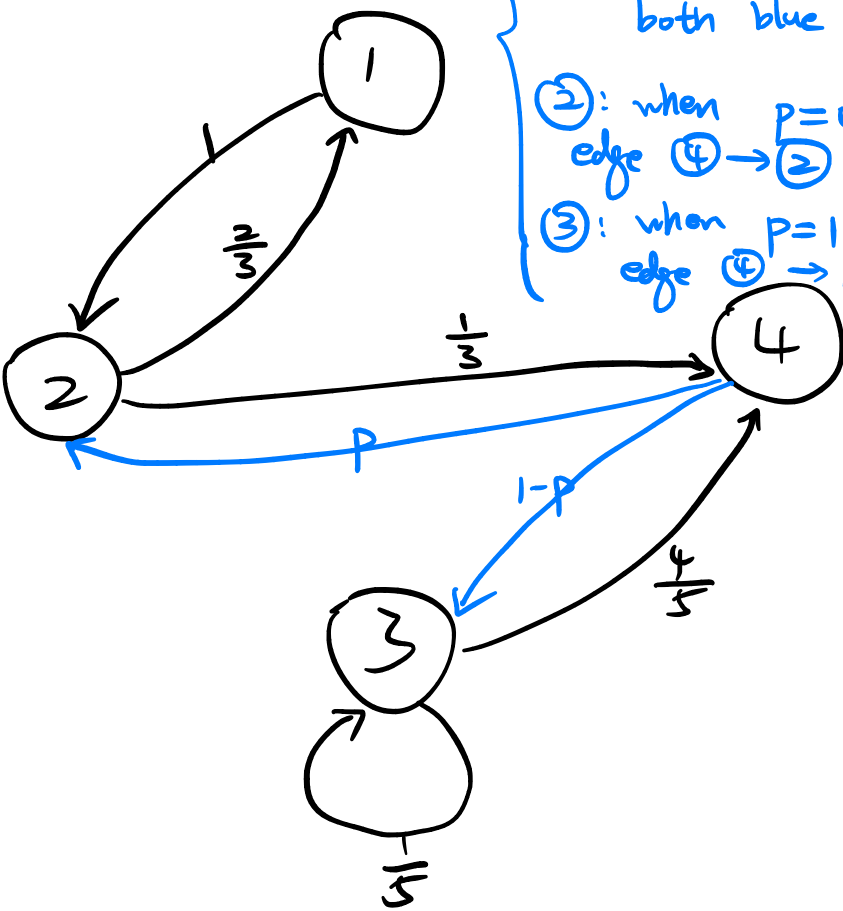
$$P=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & p & 1-p & 0 \end{pmatrix}, \quad p\in[0,1]$$

(a): Provide transition graph.

① : when $p\in(0,1)$, both blue edges appear.

② : when $p=0$ the edge ④ → ② not exist.

③ : when $p=1$ the edge ④ → ③ not exist.



for any two states in the same comm class, they can reach each other.

(b): Determine all communication classes, which classes are closed?

a comm class is closed if $\forall$ state in the commu class, it can never reach states out of the commu class.

Pf: If $p \in (0,1)$, since every two states can reach each other, com class is $\{1,2,3,4\}$, closed.

If $p=0$, two comm class $\{1,2\}$ and $\{3,4\}$. ← closed

not closed (since 2 can reach 4)

If $p=1$, two comm class $\{1,2,4\}$ and $\{3\}$.

closed

not closed (since 3 can reach 4)

(c): Now fix $p=0$. Initial dist is $\alpha^T = (0, 0, \frac{5}{9}, \frac{4}{9})$, compute dist of X_1, and dist of X_{103}.

Pf: Dist of X_1 is $\alpha^T \cdot P$

$$\underbrace{\left(0, 0, \frac{5}{9}, \frac{4}{9}\right)}_{\alpha^T} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & \frac{5}{9} & \frac{4}{9} \end{pmatrix}}_{\alpha^T}$$

$\Rightarrow \alpha^T$ is the stationary distribution.

Dist of X_{103} is $\alpha^T \cdot P^{103}$

$$= (\underbrace{\alpha^T P}_{\alpha^T}) \cdot P^{102} = \alpha^T \cdot P^{102} = \cdots = \alpha^T \cdot P$$

$$= \alpha^T = \left(0 \quad 0 \quad \frac{5}{9} \quad \frac{4}{9}\right).$$