2.2. $X \sim P(1)$, $Y \sim G(\frac{1}{2})$

(a): $\mathbb{P}(X+Y \geq 9) \leq \frac{\mathbb{E}(X+Y)}{9} = \frac{\mathbb{E}X + \mathbb{E}Y}{9}$

$\mathbb{E}X = 1$, $\mathbb{E}Y = \frac{1}{\frac{1}{2}} = 2$, $\mathbb{E}(X+Y) = 1+2 = 3$

(b): $\mathbb{P}(X+Y \geq 9) = \mathbb{P}\Big((X+Y) - \mathbb{E}(X+Y) \geq 6\Big)$

$\mathbb{P}(|X - \mathbb{E}X| \geq a) \leq \frac{\operatorname{Var} X}{a^2}$

$\leq \mathbb{P}\Big(\big|(X+Y) - \mathbb{E}(X+Y)\big| \geq 6\Big)$

$\leq \frac{\operatorname{Var}(X+Y)}{6^2} \overset{\text{indep}}{=} \frac{\operatorname{Var}(X) + \operatorname{Var}(Y)}{6^2}$

2.5: $X_1, X_2, \cdots$ i.i.d. $\sim P(\lambda)$

$S_n = X_1 + \cdots + X_n$ $\qquad \mathbb{P}\left(\frac{S_n}{n} \le t\right)$

$\mathbb{E} S_n = n \cdot \mathbb{E} X_1 = n\lambda$, $\mathrm{Var}(S_n) = n \cdot \mathrm{Var} X_1 = n\lambda$

By CLT, $\boxed{\frac{S_n - n\lambda}{\sqrt{n\lambda}} \xrightarrow{d} N(0,1) \quad (n \to \infty)}$

$$\mathbb{P}\left(\frac{S_n}{n} \le t\right) = \mathbb{P}(S_n \le nt) = \mathbb{P}\left(\frac{S_n - n\lambda}{\sqrt{n\lambda}} \le \frac{nt - n\lambda}{\sqrt{n\lambda}}\right)$$

$$= \mathbb{P}\Big(\underbrace{\frac{S_n - n\lambda}{\sqrt{n\lambda}}}_{\xrightarrow{d} N(0,1)} \le \sqrt{n} \cdot \frac{t-\lambda}{\sqrt{\lambda}}\Big)$$

$$= \mathbb{P}\Big(\frac{S_n - n\lambda}{\sqrt{n\lambda}} \cdot \underbrace{\frac{1}{\sqrt{n}}}_{\to 0 \ (n\to\infty)} \le \frac{t-\lambda}{\sqrt{\lambda}}\Big)$$

Since $\frac{S_n - n\lambda}{\sqrt{n\lambda}} \xrightarrow{d} N(0,1)$ $(n \to \infty)$, $\frac{1}{\sqrt{n}} \to 0$ $(n \to \infty)$,

we know $\frac{S_n - n\lambda}{\sqrt{n\lambda}} \cdot \frac{1}{\sqrt{n}} \xrightarrow{d} 0$ $(n \to \infty)$

so: $$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{n} \le t\right) = \mathbb{P}\left(0 \le \frac{t-\lambda}{\sqrt{\lambda}}\right) = \begin{cases} 0 & \lambda > t \\ 1 & \lambda < t \end{cases}$$

2.3: $X_1, \ldots, X_{997000} \overset{\text{i.i.d.}}{\sim} B(1, \frac{1}{2})$

$$S_{997000} = X_1 + \cdots + X_{997000}$$

$$P_A = \mathbb{P}(S_{997000} > 497000)$$

CLT:

$$\mathbb{E}S_{997000} = 997000 \times \mathbb{E}X_1 = \frac{997000}{2} = \mu$$

$$\operatorname{Var} S_{997000} = 997000 \times \operatorname{Var} X_1 = 997000 \times \frac{1}{2} \times \frac{1}{2} = \sigma^2$$

$$\frac{S_{997000} - \mathbb{E}S_{997000}}{\sqrt{\operatorname{Var} S_{997000}}} \xrightarrow{d} N(0,1) \quad (n \to \infty)$$

$$P_A = \mathbb{P}(S_{997000} > 497000) = \mathbb{P}\Big(\underbrace{\frac{S_{997000} - \mu}{\sigma}}_{\xrightarrow{d} N(0,1)} > \frac{497000 - \mu}{\sigma}\Big)$$

$$\approx \mathbb{P}\left(N(0,1) > \frac{497000 - \mu}{\sigma}\right)$$

$$= 1 - \Phi\left(\frac{497000 - \mu}{\sigma}\right)$$

SLLN:

$$\frac{S_n}{n} \xrightarrow{a.s.} \frac{1}{2} \quad (n \to \infty)$$

$$\frac{S_n}{n} \xrightarrow{d} \frac{1}{2} \quad (n \to \infty)$$

$$\mathbb{P}(S_{997000} > 497000) = \mathbb{P}\left(\underbrace{\frac{S_{997000}}{997000}}_{\text{close to } \frac{1}{2}} > \frac{497000}{997000}\right)$$

$$\approx \mathbb{P}\left(\frac{1}{2} > \frac{497000}{997000}\right)$$

(either 0 or 1)

$X_1, \ldots, X_{26} \sim \mathcal{P}(5)$ i.i.d.

$$S_{26} = X_1 + \cdots + X_{26}$$

$$\mathbb{P}(S_{26} < 120) \qquad \text{(CLT).}$$

$$\begin{cases} \mathbb{E} S_{26} = 26 \times \mathbb{E} X_1 = 26 \times 5 \\ \mathrm{Var}\, S_{26} = 26 \times \mathrm{Var}\, X_1 = 26 \times 5 \end{cases}$$

$\underline{\underline{\text{CLT}}}$: $$\frac{S_{26} - 26 \times 5}{\sqrt{26 \times 5}} \overset{d}{\approx} N(0,1)$$

$$\mathbb{P}(S_{26} < 120) = \mathbb{P}\left(\frac{S_{26} - 26 \times 5}{\sqrt{26 \times 5}} < \frac{120 - 26 \times 5}{\sqrt{26 \times 5}} \right)$$

$$= \Phi\left(\frac{120 - 26 \times 5}{\sqrt{26 \times 5}} \right) = \Phi(-0.88)$$

$$
\begin{array}{ccc}
 & 5000 & \\
100 & & 4900 \\
\checkmark & & (\text{w.p. } \tfrac{1}{2}\ \checkmark \quad \text{w.p. } \tfrac{1}{2}\ \times)
\end{array}
$$

$X_1, \dots, X_{4900}$ i.i.d. $\sim B(1, \frac{1}{2})$

independent

$$\boxed{X_i = \begin{cases} 1 & \text{if } i\ \checkmark \\ 0 & \text{if } i\ \times \end{cases}}$$

$S_{4900} = X_1 + \dots + X_{4900}$ is the num of votes supporting policy

$$\mathbb{P}(S_{4900} + 100 > 2500) = \mathbb{P}(S_{4900} > 2400)$$

$$\left\{ \begin{array}{l} \mathbb{E} S_{4900} = 4900 \times \frac{1}{2} \\ \operatorname{Var} S_{4900} = 4900 \times \operatorname{Var} X_1 = 4900 \times \frac{1}{2} \times \frac{1}{2} \end{array} \right.$$

$$\mathbb{P}\left(\underbrace{\frac{S_{4900} - 4900 \times \frac{1}{2}}{\sqrt{4900 \times \frac{1}{2} \times \frac{1}{2}}}}_{\overset{d}{\approx} N(0,1)} > \frac{2400 - 4900 \times \frac{1}{2}}{\sqrt{4900 \times \frac{1}{2} \times \frac{1}{2}}} \right)$$

$$\approx 1 - \Phi\left(\frac{2400 - 4900 \times \frac{1}{2}}{\sqrt{4900 \times \frac{1}{2} \times \frac{1}{2}}} \right) = \Phi(1.43)$$

$$\mathbb{P}(k_1 \le S_n \le k_2), \quad S_n = X_1 + \cdots + X_{500}$$

$$\mathbb{E}S_n = 500 \times \mathbb{E}X_1$$

$$\mathrm{Var}\,S_n = 500 \times \mathrm{Var}\,X_1$$

part 1

$$\boxed{\frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\,S_n}} \overset{d}{\sim} N(0,1)}$$

$$\mathbb{P}\left(\frac{k_1 - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\,S_n}} \le \frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\,S_n}} \le \frac{k_2 - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\,S_n}}\right)$$

$$\approx \Phi\left(\frac{k_2 - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\,S_n}}\right) - \Phi\left(\frac{k_1 - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\,S_n}}\right)$$

norm.cdf