Zoom polls

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160A Discrete-time stochastic process

120A — basics in prob & stats

Probability Space

$(\Omega, \mathcal{F}, \mathbb{P})$

Ω → sample space

$\mathcal{F}$ → sigma field (collection of all events)

$\mathbb{P}$ → probability measure

$\omega \in \Omega$

ω → sample points

throw dice $\Omega = \{1,2,3,4,5,6\}$

$A \in \mathcal{F}$ e.g. $A = \{1, 2\}$

$\uparrow$ event: get 1 or 2 when I throw a dice

$(A \subseteq \Omega)$

$\downarrow$

event

$\mathcal{F}$ properties:

$\mathbb{P}(\Omega) = 1$

① : $\phi \in \mathcal{F}$, $\Omega \in \mathcal{F}$

② : $\forall A \in \mathcal{F}$, $A^c \in \mathcal{F}$ (closed under complement)

$\boxed{\mathbb{P}(A) + \mathbb{P}(A^c) = 1}$

③ : $\forall A_1, A_2, \cdots \in \mathcal{F}$,

$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable union)

domain of $\mathbb{P}$

$$\boxed{\begin{aligned} \mathbb{P} : \mathcal{F} &\rightarrow [0, 1] \\ A &\mapsto \mathbb{P}(A) \end{aligned}}$$

$\mathcal{F}$: closed under complement, intersection, union

$A \cap B = (A^c \cup B^c)^c$

e.g: $\Omega = \{1,2,3\}$

$\mathcal{F} = \{\phi, \{1,2,3\}, \underline{\{1\}}\}$

not closed under complement

$\{1\}^c = \{2,3\} \notin \mathcal{F}$ $\times$ sigma-field

$\mathcal{F} = \{\phi, \{1,2,3\}, \underline{\{1\}}, \{2\}, \{1,2\}, \{3\}\}$

$\{1\}^c = \underline{\{2,3\}} \notin \mathcal{F}$ $\times$ sigma-field.

$\mathcal{F} = \{\phi, \{1,2,3\}, \{1\}, \{2\}, \{1,2\}, \{3\}, \{2,3\}, \{1,3\}\}$ ✓ sigma-field

= power set of Ω

$\mathcal{F} = \{\phi, \{1,2,3\}, \{1\}, \{2,3\}\}$ ✓ sigma-field discrete / cts r.v. X

discrete $\to$ pmf $\mathbb{P}(X=k)$

cts $\to$ pdf $f_X(x)$

$$\int_a^b f_X(x)\,dx = \mathbb{P}(X\in(a,b))$$

def of pdf

pmf $\to$ cdf, cdf $\xrightarrow{\text{difference}}$ pmf

pdf $\to$ cdf, cdf $\xrightarrow{\text{diff}}$ pdf

cdf

$$F_X(x) \overset{\text{def}}{=} \mathbb{P}(X\le x)$$

expectation : $\mathbb{E}X$

$\mathbb{E}X \xrightarrow{\text{discrete}} \sum_k k\cdot \mathbb{P}(X=k)$

$\mathbb{E}X \xrightarrow{\text{cts}} \int_{\mathbb{R}} x\cdot f_X(x)\,dx$

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

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e.g.: X cts r.v. pdf $f_X(t) = \begin{cases} c\cdot(4t-2t^2) & 0<t<2 \\ 0 & \text{else} \end{cases}$

($c>0$ constant)

(a): Find value of c

$$\int_0^2 f_X(t)\,dt = 1$$

$$c\cdot\int_0^2 4t-2t^2\,dt = 1$$

$$c\cdot\left(2t^2-\frac{2}{3}t^3\Big|_{t=0}^{2}\right)=1 \Rightarrow c=\frac{3}{8}.$$

(b): Find cdf of X

$$F_X(x) = \underline{\mathbb{P}(X\le x)} = \begin{cases} 0 & x<0 \\ \frac{3}{4}x^2-\frac{1}{4}x^3 & 0\le x\le 2 \\ 1 & x>2 \end{cases}$$

$0\le X\le x$

$$\underline{\int_0^x f_X(t)\,dt} = c\int_0^x (4t-2t^2)\,dt = \underbrace{\frac{3}{4}x^2-\frac{1}{4}x^3}$$

(c): $\mathbb{P}(X=1) = 0$

X cts r.v.

$$\mathbb{P}(\underbrace{X>1}_{X\in(1,2)}) = \int_1^2 f_X(t)\,dt$$

(d): $\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$$\begin{cases} \mathbb{E}X = \int_0^2 t\cdot f_X(t)\,dt \\ \mathbb{E}X^2 = \int_0^2 t^2\cdot f_X(t)\,dt \end{cases}$$

set $g(x)=x^2$

Recall:

$$\mathbb{E}\,g(X) = \int_{\mathbb{R}} g(x)\cdot f_X(x)\,dx$$

Independence

Def: A, B are events, they are **independent**

iff $\mathbb{P}(\underbrace{AB}_{A\cap B}) = \mathbb{P}(A)\cdot\mathbb{P}(B)$

A, B are **disjoint** / **mutually exclusive** if $A\cap B=\emptyset$

e.g: If we have A, B exclusive & independent

$$\begin{cases} A\cap B=\emptyset \Rightarrow \mathbb{P}(AB)=0 \\ \mathbb{P}(AB)=\mathbb{P}(A)\cdot\mathbb{P}(B) \end{cases}$$

$$\boxed{\mathbb{P}(A)\cdot\mathbb{P}(B)=0} \Rightarrow \mathbb{P}(A)=0 \text{ or } \mathbb{P}(B)=0$$

e.g: $X\sim U(0,1)$

$A:\{X=\frac{1}{2}\}$ $\quad B:\{X=\frac{1}{3}\}$

A, B disj, $\mathbb{P}(A)=0$, $\mathbb{P}(B)=0$

$\mathbb{P}(AB)=0=\mathbb{P}(A)\cdot\mathbb{P}(B)$, independent ✓

Conditional Probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

event condition

If A, B independent, (throw away condition)

$$\mathbb{P}(AB) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \mathbb{P}(B) \quad (\text{assume } \mathbb{P}(A) \neq 0)$$

$$\| $$

$$\mathbb{P}(B|A)$$