Markov Chain with discrete time and discrete state space. $n \in \{0, 1, 2, \cdots\}$

the value taken by MC is at most countably many, state space S is countable

$\{X_n\}$ is Markov chain, Markov property:

$$\mathbb{P}(\underbrace{X_{n+1} = i_{n+1}}_{\text{future}} \mid \underbrace{X_n = i_n}_{\text{present}}, \underbrace{X_{n-1} = i_{n-1}, \cdots, X_0 = i_0}_{\text{past}})$$

$$= \mathbb{P}(\underbrace{X_{n+1} = i_{n+1}}_{\text{future}} \mid \underbrace{X_n = i_n}_{\text{present}})$$

$\Downarrow$

Transition probability matrix P.

$$P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

transition from i to j (one-step)

$$(P^k)_{ij} = \mathbb{P}(X_{n+k} = j \mid X_n = i)$$

prob transition from i to j (k-step) e.g.: $\{X_n\}$ MC, state space $S=\{1,2,3\}$.

$$P = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

initial dist $\alpha^T = (0.1, 0.6, 0.3)$. Compute.

(a): $\mathbb{P}(X_7=1 \mid X_6=3) = P_{31} = \boxed{0.3}$

(b): $\mathbb{P}(\underbrace{X_9=3}_{\text{future}} \mid \underbrace{X_1=1, X_5=2}_{\text{past}}, \underbrace{X_7=1}_{\text{present}})$

$\overset{\text{Markov property}}{=} \mathbb{P}(X_9=3 \mid X_7=1)$ (two-step transition prob from 1 to 3)

$= (P^2)_{1,3} = \boxed{0.12}$

$$P^2 = \begin{pmatrix} 0.27 & 0.61 & 0.12 \\ 0.22 & 0.46 & 0.32 \\ 0.17 & 0.31 & 0.52 \end{pmatrix}$$

(c): $\mathbb{P}(X_0=2 \mid X_1=3)$

Apply Bayes thm $= \dfrac{\mathbb{P}(X_1=3 \mid X_0=2)\cdot \mathbb{P}(X_0=2)}{\mathbb{P}(X_1=3)}$

$$= \frac{P_{23} \cdot 0.6}{\mathbb{P}(X_1=3)}$$

dist of X_1 is $\alpha^T \cdot P$

$$\mathbb{P}(X_1=j) = \sum_i \underbrace{\mathbb{P}(X_0=i)}\cdot \underbrace{\mathbb{P}(X_1=j \mid X_0=i)}$$

$$= \sum_i (\alpha^T)_i \cdot P_{ij} = (\alpha^T \cdot P)_j$$

$$\alpha^T \cdot P = (0.1 \quad 0.6 \quad 0.3)\cdot \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

$$\mathbb{P}(X_1=3) = 0.1\times 0.8 + 0.6\times 0.4 + 0.3\times 0$$

$$= 0.08 + 0.24 = 0.32$$

$$\mathbb{P}(X_0=2 \mid X_1=3) = \frac{0.4\times 0.6}{0.32} = \frac{24}{32} = \boxed{\frac{3}{4}}$$

(d): $\mathbb{E}X_2$

dist of X_2 is given by $\alpha^T \cdot P^{\boxed{2}}$ transit twice from X_0 to X_2.

$$\alpha^T \cdot P^2 = \begin{pmatrix} 0.1 & 0.6 & 0.3 \end{pmatrix} \begin{pmatrix} 0.27 & 0.61 & 0.12 \\ 0.22 & 0.46 & 0.32 \\ 0.17 & 0.31 & 0.52 \end{pmatrix}$$

$$= (\underbrace{0.21}_{\mathbb{P}(X_2=1)}, \underbrace{0.43}_{\mathbb{P}(X_2=2)}, \underbrace{0.36}_{\mathbb{P}(X_2=3)})$$

$$\mathbb{E}X_2 = \sum_{i=1}^{3} i \cdot \mathbb{P}(X_2 = i)$$

$$= 1 \times 0.21 + 2 \times 0.43 + 3 \times 0.36 = \boxed{2.15}$$

e.g.: P always stochastic matrix (its row always sum up to 1)

P is called doubly stochastic if its columns also sum up to 1. (each row & each column sum up to 1)

Let $\{X_n\}$ be MC with state space $S=\{1,2,\cdots,k\}$, and its one-step transition matrix P is doubly stochastic, its initial dist is uniform on S.

Show that $\forall n$, X_n is always uniform on S.

Pf: Apply induction,

1: For $n=0$, X_0 is uniform on S. ✓

2: Assume X_k is uniform on S, want to prove X_{k+1} is uniform on S.

$\forall i, \mathbb{P}(X_{k+1} = i)$ (law of total prob)

$$= \sum_{j=1}^{k} \underbrace{\mathbb{P}(X_k = j)}_{\frac{1}{k}} \cdot \underbrace{\mathbb{P}(X_{k+1} = i \mid X_k = j)}_{P_{ji}}$$

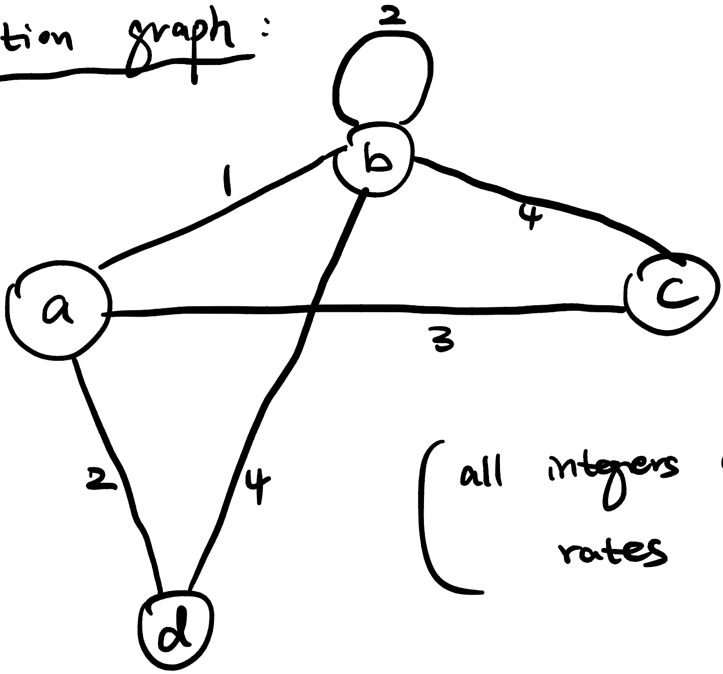
(induction assumption)

$$= \frac{1}{k} \underbrace{\sum_{j=1}^{k} P_{ji}}_{\substack{\text{sum of} \\ \text{the } i\text{-th column} \\ \text{of } P = 1 \\ \text{(doubly stochastic)}}} = \frac{1}{k} \Rightarrow X_{k+1} \text{ is uniform on } S.$$

finishes the induction. ✓

Transition graph:

e.g.:



(all integers are rates)

what is $P_{ad} = \dfrac{2}{1+3+2} = \dfrac{2}{6}$

(normalize rates $\Rightarrow$ transition prob) $= \dfrac{1}{3}$

$$P_{bc} = \frac{4}{1+4+4+2} = \frac{4}{11}$$

$$P_{ij} = \frac{r_{ij}}{\sum_{k \in S} r_{ik}}$$