e.g.: (Betrand's Ballot Problem)

$\begin{cases} A \to \text{receives 200 votes} \\ B \to \text{receives 100 votes} \end{cases}$ > 300 voters

assume each voter i.i.d. $\frac{1}{2}$ votes for A w.p. $\frac{1}{2}$ votes for B. What's the prob A is always ahead throughout the count?

Sol:

Let S_n be the number of votes A gets − the number of votes B gets at the moment the n-th voter votes.

Since all voters are i.i.d., $S_n = \sum_{i=1}^{n} \xi_n$ with increments $\xi_1, \xi_2, \text{———}$ i.i.d.

If the i-th voter votes for A $\Rightarrow \xi_i = 1$

If the i-th voter votes for B $\Rightarrow \xi_i = -1$

So $\mathbb{P}(\xi_i=1)=\mathbb{P}(\xi_i=-1)=\frac{1}{2}.$

$$\Downarrow$$

$\{S_n\}$ is SSRW

$\mathbb{P}(A \text{ is always ahead}) = \mathbb{P}(S_n>0 \text{ for } \forall n\geq 1)$

time $n\in\{1,2,\cdots,300\}$, we know $S_{300}=200-100=100$

condition

Actually want to calculate

$$\mathbb{P}(\forall\ 1\leq n\leq 300,\ S_n>0 \mid S_{300}=100)$$

$$=\frac{\mathbb{P}(\forall 1\leq n\leq 300, S_n>0, S_{300}=100)}{\mathbb{P}(S_{300}=100)}$$

$$=\frac{\#\text{ of SSRW paths that is always positive and ends at }(300,100)}{\#\text{ of SSRW paths that ends at }(300,100)}$$

For SSRW, all paths have same probability of appearing

First count # of SSRW paths that ends at $(300, 100)$

$$\begin{cases} 300 \text{ increments} \\ \text{They add up to } 100 \end{cases} \Rightarrow$$ assume k increments take value 1, $300-k$ incr take value -1

$$\text{sum} = k \cdot 1 + (300-k) \cdot (-1) = 2k - 300 = 100$$

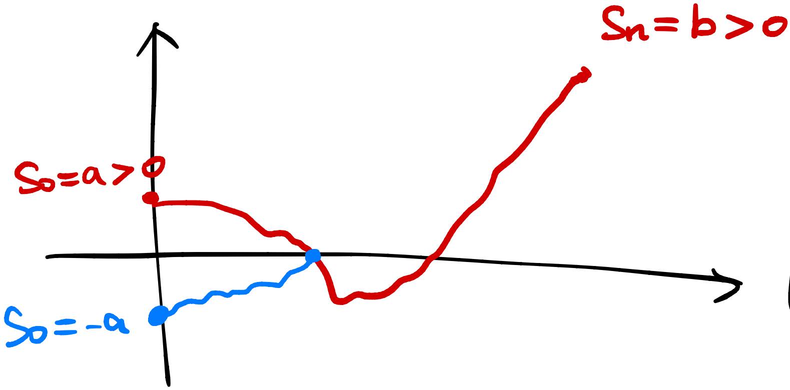
$$\Downarrow$$

$$\underline{k = 200}$$

so # of paths: $\underline{\underline{\binom{300}{200}}}$

Then, count # of paths that are always positive and ends at (300, 100).

For such path $S_1 = 1$



count # of paths that never hits 0.

# of paths with $S_0 = a > 0$, ends at $S_n = b > 0$ and has hit 0 in between

$$\|$$

# of paths with $S_0 = -a < 0$, ends at $S_n = b > 0$

Reflection Principle

☆ Important Trick.

Consider complement,

count # of paths that has hit 0 and ends at (300, 100), with $\boxed{S_1 = 1}$

By reflection principle,

this # = # of paths that starts at $S_1 = -1$ and ends at $S_{300} = 100$

$\Downarrow$

299 increments, add up to 101

$\Downarrow$

$$\begin{cases} 200 \text{ incr taking value } 1 \\ 99 \text{ incr taking value } -1 \end{cases}$$

$\Downarrow$

$$\binom{299}{200}$$

the # of paths that starts at $S_1 = 1$, ends at $S_{300} = 100$

$\Downarrow$

299 incr, add up to 99

$\Downarrow$

199 incr value 1, 100 incr value -1

$\Downarrow$

$$\binom{299}{199}$$

$$\text{numerator} = \binom{299}{199} - \binom{299}{200}$$

$\parallel$

# of paths ends at $S_{300} = 100$ and stays positive.

$$\mathbb{P}(\sim\sim) = \frac{\binom{299}{199} - \binom{299}{200}}{\binom{300}{200}}$$

$$= \frac{\frac{299!}{199!\,100!} - \frac{299!}{200!\,99!}}{\frac{300!}{200!\,100!}}$$

$$= \frac{\frac{200\cdot 299! - 100\cdot 299!}{\cancel{200!}\ \cancel{100!}}}{\frac{300!}{\cancel{200!}\ \cancel{100!}}}$$

$$= \frac{299! \times 100}{300!} = \frac{100}{300} = \boxed{\frac{1}{3}}$$

Remark :

$a = 200, \quad b = 100,$

$$\mathbb{P}(A \text{ is always ahead}) = \frac{a-b}{a+b}$$

$$= \frac{200-100}{200+100} = \boxed{\frac{1}{3}}$$