$\{S_n\}$ SSRW, $S_0 = 0$, $S_n = \underbrace{\xi_1 + \cdots + \xi_n}_{\xi_1,\, \xi_2,\, \cdots \text{ i.i.d.}}$

simple symmetric random walk

$$\begin{cases} \mathbb{P}(\xi_1 = 1) = \frac{1}{2} \\ \mathbb{P}(\xi_1 = -1) = \frac{1}{2} \end{cases}$$

e.g.: $\{S_n\}$ SRW (asymmetric), $S_0 = 0$,

$p = 0.4$, $q = 0.6$

p, q

$$\begin{cases} \mathbb{P}(\xi_1 = 1) = p \\ \mathbb{P}(\xi_1 = -1) = q \end{cases}$$

(a): $\mathbb{P}(S_4 = 2,\ S_m \neq 3 \text{ for all } m \in \{1,2,3\})$

from time 0 to 4, there are 4 increments $\xi_1, \xi_2, \xi_3, \xi_4$, should be $\begin{cases} 3 \text{ ups} \\ 1 \text{ downs} \end{cases}$

$\boxed{\text{UUDU, UDUU, DUUU,}}$ $\underline{\text{UUUD}} \Rightarrow S_3 = 3$ ✗

$\mathbb{P}(\sim\!\sim) = 3 \times p^3 \times q = \underline{3 \times 0.4^3 \times 0.6}$

(b): $\mathbb{P}(S_2=2,\ S_5=1)$ $\Rightarrow$ consider joint dist of (S_m, S_n)

☆: S_2 is not independent of S_5,

$S_5 = S_2 + \xi_3 + \xi_4 + \xi_5$

$\Downarrow$

$(S_2 = \xi_1 + \xi_2)$

☆: S_2, ξ_3, ξ_4, ξ_5 independent

Condition on S_2,

$$= \underbrace{\mathbb{P}(S_2=2)}_{\mathbb{P}(\xi_1+\xi_2=2)} \cdot \underbrace{\mathbb{P}(S_5=1 \mid S_2=2)}$$

$\mathbb{P}(\xi_1+\xi_2=2)$

$\parallel$

$\mathbb{P}($both take value $1)$

$= p^2$

$= \mathbb{P}(S_2+\xi_3+\xi_4+\xi_5=1 \mid S_2=2)$

$= \mathbb{P}(\xi_3+\xi_4+\xi_5=-1 \mid S_2=2)$

independent

$= \mathbb{P}(\xi_3+\xi_4+\xi_5=-1)$

$= \mathbb{P}($ 2 of them are -1, 1 of them is $1)$

$= \binom{3}{1} \times p \times q^2$

not i.i.d.

$$\begin{cases} S_n: \text{value of SRW at time } n \\ S_n - S_{n-1} = \xi_n: \text{increment of SRW at time } n \\ \qquad\qquad\qquad\qquad \text{i.i.d.} \end{cases}$$

Whenever we consider joint dist of (S_m, S_n) $(m<n)$, we always condition on S_m

(past, value of SRW at an earlier time)

S_m is independent of $\xi_{m+1}, \xi_{m+2}, \cdots$

(c): $\mathbb{P}(S_2=2,\ S_4=3,\ S_5=1)$

Idea: first condition on S_2, then condition on S_4.

$$= \mathbb{P}(S_2=2)\cdot \mathbb{P}(S_4=3,\ S_5=1 \mid S_2=2)$$

$$\mathbb{P}(S_4=3 \mid S_2=2) \cdot \mathbb{P}(S_5=1 \mid S_4=3, S_2=2)$$

$$\mathbb{P}(S_4=3 \mid S_2=2) = \mathbb{P}(S_4-S_2=1 \mid S_2=2) \overset{\text{independance}}{=} \mathbb{P}(S_4-S_2=1)$$

$$\mathbb{P}(S_5=1 \mid S_4=3, S_2=2) = \mathbb{P}(\underbrace{S_5-S_4}_{\xi_5}=-2 \mid \underbrace{S_4=3}_{\xi_1,\dots,\xi_4},\ \underbrace{S_2=2}_{\xi_1,\xi_2}) \overset{\text{independence}}{=} \mathbb{P}(S_5-S_4=-2)$$

e.g.: $\{S_n\}$ SSRW, compute $\mathbb{P}(S_n=y \mid S_m=x)$ for $n>m$ and $n<m$.

①: Case $n>m$. $\Rightarrow \begin{cases} S_n & \text{future} \\ S_m & \text{past} \end{cases}$

$$\mathbb{P}(S_n=y \mid S_m=x) = \mathbb{P}(S_n-S_m=y-x \mid S_m=x)$$

$\underset{\xi_{m+1}+\cdots+\xi_n}{\underset{\parallel}{S_n-S_m}}$ independent of $\underset{\xi_1+\cdots+\xi_m}{\underset{\parallel}{S_m}}$

$$= \mathbb{P}(S_n-S_m=y-x)$$

$$= \mathbb{P}(\underbrace{\xi_{m+1}+\xi_{m+2}+\cdots+\xi_n}_{n-m \text{ increments}}=y-x)$$

assume k of them take value 1,

then $n-m-k$ of them take value -1

$\Downarrow$

$$S_n-S_m=\underbrace{k-(n-m-k)}_{k\times 1+(n-m-k)\times(-1)}=2k-n+m$$

set it equal to $y-x$ $\quad k\in\{0,1,\cdots,n-m\}$

$$y-x=2k-n+m \Rightarrow \text{solve } k=\frac{y-x+n-m}{2}$$

$= \mathbb{P}($ among $n-m$ increments, k of them take value $1)$

$= \binom{n-m}{k} \cdot \left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\right)^{n-m-k}$

$= \binom{n-m}{k} \cdot \left(\frac{1}{2}\right)^{n-m}$

$\mathbb{P}(S_n = y \mid S_m = x) =$

$$\begin{cases} 0 & \text{else} \\ \binom{n-m}{\frac{n-m+y-x}{2}} \cdot \left(\frac{1}{2}\right)^{n-m} & \text{if } \frac{n-m+y-x}{2} \in \{0, 1, \dots, n-m\} \end{cases}$$

Binomial

Come from $k \in \{0, 1, \dots, n-m\}$

② : Case $n<m$

$$\mathbb{P}(S_n = y \mid S_m = x)$$

past future

$$= \frac{\mathbb{P}(S_m = x \mid S_n = y) \cdot \underbrace{\mathbb{P}(S_n = y)}}{\underbrace{\mathbb{P}(S_m = x)}}$$

just calc.

$$\mathbb{P}(S_n = y) = \mathbb{P}(\xi_1 + \cdots + \xi_n = y)$$

$\begin{cases} \text{assume } k \text{ of them are } 1 \\ n-k \text{ of them are } -1 \end{cases}$

$k-(n-k) = y \Rightarrow$ solve $k = \frac{n+y}{2}$

$k \in \{0, 1, \ldots, n\}$

So $\mathbb{P}(S_n = y) = \begin{cases} 0 & \text{else} \\ \binom{n}{\frac{n+y}{2}} \cdot \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)^{n-k} & \\ = \binom{n}{\frac{n+y}{2}} \cdot \left(\frac{1}{2}\right)^n & \text{if } \frac{n+y}{2} \in \{0, 1, \ldots, n\} \end{cases}$

$$\mathbb{P}(S_n = y \mid S_m = x)$$

$$= \begin{cases} \dfrac{\dbinom{m-n}{\frac{m-n+x-y}{2}} \cdot \dbinom{n}{\frac{n+y}{2}}}{\dbinom{m}{\frac{m+x}{2}}} & \text{if } \frac{m-n+x-y}{2} \in \{0,1,\dots,m-n\} \text{ and } \frac{m+x}{2} \in \{0,1,\dots,m\} \text{ and } \frac{n+y}{2} \in \{0,1,\dots,n\} \\ 0 & \text{else} \end{cases}$$

hypergeometric