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e.g.: 2 student typist, $\begin{cases} A \\ B \end{cases}$

$\begin{cases} \text{\# of typos of } A \sim \mathcal{P}(\lambda_A = 1) \\ \text{\# of typos of } B \sim \mathcal{P}(\lambda_B = 10) \end{cases}$

Now type a letter, work done by

$\begin{cases} A & \text{w.p. } \frac{1}{3} \\ B & \text{w.p. } \frac{2}{3} \end{cases}$

(a): $\mathbb{P}(\text{letter contain exactly 1 typo})$

discuss by case ⇒ Law of Total Prob.

$= \underbrace{\mathbb{P}(A \text{ types letter})}_{\frac{1}{3}} \cdot \mathbb{P}(\sim\sim \mid A \text{ types letter})$

$+ \underbrace{\mathbb{P}(B \text{ types letter})}_{\frac{2}{3}} \cdot \mathbb{P}(\sim\sim \mid B \text{ types letter})$

$\mathbb{P}$(exactly 1 error | A types letter)

$= \underbrace{\text{point mass at 1 of } \mathcal{P}(1)}$

$= \frac{1^1}{1!} e^{-1} = e^{-1}$

$$\boxed{\begin{gathered}\mathbb{P}(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \\ X \sim \mathcal{P}(\lambda)\end{gathered}}$$

(b): We find $\overset{\text{consequence}}{\underline{\text{the letter actually has no typos}}}$. Given this, $\mathbb{P}(\underbrace{\text{B typed letter}}_{\text{Cause}})$?

$\mathbb{P}(\text{cause} \mid \text{consequence})$

☆: Bayes

$\mathbb{P}$(B typed letter | no typo)

point mass at 0 of $\mathcal{P}(10)$

$$= \frac{\mathbb{P}(\text{no typo} \mid \text{B typed letter}) \cdot \mathbb{P}(\text{B typed})}{\mathbb{P}(\text{no typo} \mid \text{B typed}) \cdot \mathbb{P}(\text{B typed}) + \mathbb{P}(\text{no typo} \mid \text{A typed}) \cdot \mathbb{P}(\text{A typed})}$$

point mass at 0 of $\mathcal{P}(1)$

$$= \frac{\frac{10^0}{0!} e^{-10} \times \frac{2}{3}}{\frac{10^0}{0!} e^{-10} \times \frac{2}{3} + \frac{1^0}{0!} e^{-1} \times \frac{1}{3}} = \frac{2e^{-10}}{2e^{-10} + e^{-1}}$$

e.g: Joint density $f(x,y)=\frac{x+y}{4}$ $(0<x<y<2)$

(a): $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$

$$f_Y(y)=\int_0^y f(x,y)\,dx$$ (integrate joint density w.r.t. x to get marginal of Y)

$$=\int_0^y \frac{x+y}{4}\,dx=\frac{3}{8}y^2 \quad (0<y<2)$$

So $$f_{X|Y}(x|y)=\frac{\frac{x+y}{4}}{\frac{3}{8}y^2}=\frac{2}{3}\left(\frac{x}{y^2}+\frac{1}{y}\right)$$
$(0<x<y<2)$

(b): Calculate $\mathbb{P}(X<\frac{1}{2}|Y=1)$, $\mathbb{P}(X<\frac{3}{2}|Y=1)$

Since the condition is $Y=1$, we just need to take $y=1$ in $f_{X|Y}(x|y)$ to get

$$f_{X|Y}(x|1)=\frac{2}{3}(x+1) \quad (0<x<1)$$

As a result,

$$\mathbb{P}(X<\tfrac{1}{2}|Y=1)=\int_0^{\frac{1}{2}} f_{X|Y}(x|1)\,dx$$
$$=\int_0^{\frac{1}{2}}\frac{2}{3}(x+1)\,dx=\frac{5}{12}$$

$$\mathbb{P}(X<\tfrac{3}{2} \mid Y=1) = \int_0^{\frac{3}{2}} f_{X|Y}(x|1)\,dx = 1$$

since $f_{X|Y}(x|1)$ has its support on $(0,1)$

(c): Calculate $\mathbb{E}(X^2 \mid Y=y)$ and verify

$$\int_{-\infty}^{+\infty} \mathbb{E}(X^2 \mid Y=y)\cdot f_Y(y)\,dy = \mathbb{E}X^2$$

$$\begin{aligned}
\mathbb{E}(X^2 \mid Y=y) &= \int_0^y x^2 \cdot f_{X|Y}(x|y)\,dx \\
&= \int_0^y x^2 \cdot \frac{2}{3}\left(\frac{x}{y^2}+\frac{1}{y}\right)dx \\
&= \frac{2}{3y^2}\int_0^y x^3\,dx + \frac{2}{3y}\cdot\int_0^y x^2\,dx \\
&= \frac{7}{18}y^2
\end{aligned}$$

So

$$\text{LHS} = \int_0^2 \frac{7}{18}y^2\cdot\frac{3}{8}y^2\,dy = \frac{7}{48}\int_0^2 y^4\,dy$$

$$= \frac{14}{15}$$

$$\text{RHS} = \mathbb{E}X^2 = \int_0^2 x^2\cdot f_X(x)\,dx$$

$$f_X(x) = \int_x^2 f(x,y)\,dy = \int_x^2 \frac{x+y}{4}\,dy = \frac{x+1}{2} - \frac{3}{8}x^2$$

(derive $f_X(x)$) $\qquad (0<x<2)$

So $$\text{RHS} = \int_0^2 x^2\left(\frac{x+1}{2} - \frac{3}{8}x^2\right)dx$$

$$= \frac{1}{2}\int_0^2 x^3\,dx + \frac{1}{2}\int_0^2 x^2\,dx - \frac{3}{8}\int_0^2 x^4\,dx$$

$$= \frac{14}{15}$$

it's proved.

<u>Remark</u>: The equation we proved in (c) is actually the law of iterated expectation

$$\mathbb{E}\left[\mathbb{E}(X^2|Y)\right] = \mathbb{E}X^2$$

to see why $\int \mathbb{E}(X^2|Y=y)\cdot f_Y(y)\,dy = \mathbb{E}\left[\mathbb{E}(X^2|Y)\right]$, Denote $\mathbb{E}(X|Y) = h(Y)$ as a function of Y (a random variable),

recall how we got $\mathbb{E}(X|Y)$, by calculating $\mathbb{E}(X|Y=y) = h(y)$ and then replace y with Y.

e.g.: If we know

$$\mathbb{E}(X|Y=y) = y^2$$

then

$$\mathbb{E}(X|Y) = Y^2$$

So $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}\, h(Y)$

$= \int h(y) \cdot f_Y(y)\, dy$ (formula for expectation)

$= \int \mathbb{E}(X|Y=y) \cdot f_Y(y)\, dy$ ✓

CLT:

$X_1, \cdots, X_n, \cdots$ i.i.d.

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$\mathbb{E}S_n = n \cdot \mathbb{E}X_1$$

$$\mathrm{Var}(S_n) = n \cdot \mathrm{Var}(X_1)$$

Standardization:

$$\frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - n \cdot \mathbb{E}X_1}{\sqrt{n} \cdot \sqrt{\mathrm{Var}(X_1)}}$$

$$\xrightarrow{d} N(0,1) \quad (n \to \infty)$$