PSTAT 8 Sample Final Answer

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Theorem 1 (Problem 1)**.** *For four sets A, B, C, D, $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.*

Proof. Use direct proof. By the definition of subset relationship, we have to prove that $\forall x \in (A \times B) \cup (C \times D)$, such x satisfies $x \in (A \cup C) \times (B \cup D)$.

Notice that the elements in $A \times B$ and $C \times D$ are ordered pairs, so $\forall x \in (A \times B) \cup (C \times D)$, x has the form as an ordered pair that $x = (x_1, x_2)$. We know by the definition of set union that $x \in A \times B$ or $x \in C \times D$. By the definition of the Cartesian product, $x_1 \in A$ and $x_2 \in B$ or $x_1 \in C$ and $x_2 \in D$.

For the first case where $x_1 \in A$ and $x_2 \in B$, $x_1 \in A \subset A \cup C, x_2 \in B \subset B \cup D$ so $x = (x_1, x_2) \in (A \cup C) \times (B \cup D)$.

For the second case where $x_1 \in C$ and $x_2 \in D$, $x_1 \in C \subset A \cup C, x_2 \in D \subset B \cup D$ so $x = (x_1, x_2) \in (A \cup C) \times (B \cup D)$.

As a result, for both possible cases it must be true that $x \in (A \cup C) \times (B \cup D)$, so the conclusion is proved. □

Theorem 2 (Problem 2)**.** *There are six positive integers $x_1, ..., x_6$, prove that at least two of them will have the same remainder when divided by 5.*

Proof. Prove by contradiction.

Assume that any two of them will not have the same remainder when divided by 5. Denote $r_1, ..., r_6$ as the respective remainder of $x_1, ..., x_6$ divided by 5 so $r_1, ..., r_6$ can only take values in $\{0, 1, 2, 3, 4\}$. By our assumption, $r_1, ..., r_6$ are six different values so $\{r_1, ..., r_6\}$ has cardinality six while $\{r_1, ..., r_6\} \subset \{0, 1, 2, 3, 4\}$ has cadinality five, a contradiction! □

Remark. *One may also solve this problem by constructing a function $f : \{x_1, ..., x_6\} \to \{0, 1, 2, 3, 4\}$ such that $f(x) = x \mod 5$ (the remainder of x divided by 5). If there exists $i \neq j$ such that $x_i = x_j$, it's the trivial case and the original statement is obviously true.*

Otherwise, $x_1, ..., x_6$ are distinct integers. Since the domain and the codomain are both finite sets and the codomain (cardinality 5) contains strictly less elements than the domain (cardinality 6), such f cannot be injective, so the original statement is proved.

The pigeonhole principle *says that if n items are put into m containers, there must exist at least one container that contains at least $\lceil \frac{n}{m} \rceil$ items where $\lceil \frac{n}{m} \rceil$ is the smallest integer that is larger or equal to $\frac{n}{m}$ (one may try to prove it by contradiction). Problem 2 is a special case of the pigeonhole principle since by taking $n = 6, m = 5$, we see that $\lceil \frac{n}{m} \rceil = 2$ so there must exist a number $r \in \{0, 1, 2, 3, 4\}$ such that at least 2 of the integers in $x_1, ..., x_6$ have the same remainder r divided by 5.*

Theorem 3 (Problem 3)**.** *Define a relationship R on $\mathbb{R}^2$ that $(a, b)R(c, d)$ if and only if $a^2 + b^2 < c^2 + d^2$. Is this an equivalence relationship?*

Proof. By the definition of equivalence relationship, we have to check whether R is reflexive, symmetric and transitive.

First notice that $\forall (a, b) \in \mathbb{R}^2, a^2 + b^2 < a^2 + b^2$ is not true. So we make the judgement that it cannot be an equivalence relationship. In order to prove it, we just have to raise a counterexample.

Consider $(0, 0) \in \mathbb{R}^2$, $0 = 0^2 + 0^2 < 0^2 + 0^2 = 0$ is not true, so $(0, 0)R(0, 0)$ is not true and R is not reflexive. As a result, R is not an equivalence relationship.

□

Theorem 4 (Problem 4)**.** *Consider function* $f : \mathbb{R} \to (0, \infty)$ *such that* $f(x) = e^x$*, prove that it's bijective.*

Proof. Use direct proof. In order to prove it's bijective, we have to prove that it's injective and surjective. **In the following proof,** log **actually means** ln**, the natural logarithm.**

To prove that it's injective, $\forall x, y \in \mathbb{R}$, if $f(x) = f(y)$, then $e^x = e^y$, so by taking logarithm on both sides, we see $x = \log(e^x) = \log(e^y) = y$. This proves that f is injective.

To prove it's surjective, $\forall a \in (0, \infty)$, consider $f(x) = a$ so $e^x = a$, we will see that $x = \log a \in \mathbb{R}$ since a is a positive real number and logarithm is defined for positive real number. So $\forall a \in (0, \infty), \exists x = \log a \in \mathbb{R}$ such that $f(x) = a$. This proves that f is surjective.

□

Theorem 5 (Problem 5)**.** *Consider function* $f : \mathbb{Z} \to \mathbb{Z}$ *defined as* $f(n) = n^2 + 3$*, is it injective, surjective, bijective?*

Proof. First check injectivity. $\forall m, n \in \mathbb{Z}$, if $f(m) = f(n)$, then $m^2 + 3 = n^2 + 3$ so $m^2 = n^2$ and $m = \pm n$. This is telling us that $m = n$ is not necessarily true and we can make the judgement that f is not injective.

To prove that it's not injective, we just have to raise a counterexample. Consider $m = 1, n = -1$ so $m, n \in \mathbb{Z}, m \neq n$ but $f(m) = f(1) = 1^2 + 3 = 4 = (-1)^2 + 3 = f(-1) = f(n)$ so f is not injective.

Now check surjectivity. $\forall a \in \mathbb{Z}$, consider $f(n) = a$ so $n^2 + 3 = a$ and $n^2 = a - 3$. It's quite obvious that $a - 3$ can be negative but n^2 is always non-negative. We can make the judgement that f is not surjective.

To prove that it's not surjective, we just have to raise a counterexample. Consider $a = 0 \in \mathbb{Z}$ and if $\exists n \in \mathbb{Z}$ such that $f(n) = a$, then $n^2 = a - 3 = -3 < 0$ which is impossible for any integer n. So $a = 0$ is in the codomain but not in the range and f is not surjective.

Since f is not injective, it cannot be bijective. □

Theorem 6 (Problem 6)**.** *For sets* $A_1, ..., A_n$*, first prove that* $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ *and then prove that* $\forall n \in \mathbb{Z}, n \geq 2, \left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c$

Proof. Use direct proof. To prove that two sets are equal, just need to prove the subset relationship in both directions.

Let's first prove $(A_1 \cup A_2)^c \subset A_1^c \cap A_2^c$. $\forall x \in (A_1 \cup A_2)^c$, x is not an element in $A_1 \cup A_2$. Since $A_1 \subset A_1 \cup A_2, A_2 \subset A_1 \cup A_2$, we know that $x \notin A_1$ and $x \notin A_2$ so $x \in A_1^c$ and $x \in A_2^c$ so $x \in A_1^c \cap A_2^c$, proved.

Then prove $A_1^c \cap A_2^c \subset (A_1 \cup A_2)^c$. $\forall x \in A_1^c \cap A_2^c$, by the definition of set intersection, $x \in A_1^c$ and $x \in A_2^c$ so $x \notin A_1$ and $x \notin A_2$. Since $A_1 \subset A_1 \cup A_2, A_2 \subset A_1 \cup A_2$, we know that $x \notin A_1 \cup A_2$ so $x \in (A_1 \cup A_2)^c$, proved.

So we have proved that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. Now let's prove the equation for n sets by mathematical induction.

When $n = 2$, $LHS = (A_1 \cup A_2)^c, RHS = A_1^c \cap A_2^c$ so both sides are equal by the proof above.

Now assume that the conclusion is true for $n = k$, i.e. $\left(\bigcup_{i=1}^k A_i\right)^c = \bigcap_{i=1}^k A_i^c$. When $n = k + 1$,

$$LHS = \left(\bigcup_{i=1}^{k+1} A_i\right)^c = \left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)^c = \left(\bigcup_{i=1}^{k} A_i\right)^c \cap A_{k+1}^c \tag{1}$$

$$= \left(\bigcap_{i=1}^{k} A_i^c\right) \cap A_{k+1}^c = \bigcap_{i=1}^{k+1} A_i^c = RHS \tag{2}$$

for the last equality on the first line, we view $\left(\bigcup_{i=1}^{k} A_i\right)$ as a whole term and use the conclusion that $(A \cup B)^c = A^c \cap B^c$ we have proved above and for the first equality on the second line, we use the induction assumption. So we have proved that the conclusion holds for $n = k + 1$ and this completes the induction process, proved.

□

Theorem 7 (Problem 7). *Prove by induction that* $\forall n \in \mathbb{Z}, n \geq 0, 9|(4^{3n} + 8)$.

Proof. Prove by mathematical induction.

When $n = 0$, $r = 4^{3n} + 8 = 1 + 8 = 9$ so $9|9$ is true.

Let's assume that this conclusion holds for $n = k$, i.e. $9|(4^{3k} + 8)$. When $n = k + 1$, the number becomes $4^{3(k+1)} + 8 = 4^{3k+3} + 8 = 4^3 4^{3k} + 8 = 64 \times 4^{3k} + 8$. Notice that $64 \times 4^{3k} + 8 = 64 \times (4^{3k} + 8) - 64 \times 8 + 8 = 64 \times (4^{3k} + 8) - 63 \times 8$ where by induction assumption $9|(4^{3k} + 8)$ so $9|64 \times (4^{3k} + 8)$ and since $9|63$, we know $9|63 \times 8$. As a result, $9|[64 \times (4^{3k} + 8) - 63 \times 8]$ so $9|[64 \times 4^{3k} + 8]$ so $9|(4^{3(k+1)} + 8)$. When $n = k + 1$ the conclusion still holds, this completes the induction process, proved. □

Theorem 8 (Problem 8). *Show by induction that for any positive integer n,*

$$\sum_{j=1}^{n} \frac{j}{(j+1)!} \leq 1 - \frac{1}{(n+1)!} \tag{3}$$

Proof. Prove by mathematical induction.

When $n = 1$, $LHS = \frac{1}{2!} = \frac{1}{2}, RHS = 1 - \frac{1}{2!} = \frac{1}{2}$ so $LHS \leq RHS$, conclusion holds.

Let's assume that this conclusion holds for $n = k$, i.e. $\sum_{j=1}^{k} \frac{j}{(j+1)!} \leq 1 - \frac{1}{(k+1)!}$. When $n = k + 1$,

$$LHS = \sum_{j=1}^{k+1} \frac{j}{(j+1)!} = \sum_{j=1}^{k} \frac{j}{(j+1)!} + \frac{k+1}{(k+2)!} \tag{4}$$

$$\leq 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{k+2-(k+1)}{(k+2)!} = 1 - \frac{1}{(k+2)!} = RHS \tag{5}$$

where on the second line the inequality comes from induction assumption. This completes the induction process and it's proved. □

Theorem 9 (Problem 9). *Let* $\log_2 n$ *denote the log with base 2 and* $\log^{(k)} n$ *is iteratively defined as* $\log^{(k-1)}(\log_2 n)$ *if it's well-defined and with initial value* $\log^{(0)} n = n$ *for non-negative integers* k, n. *Define the iterated logarithm as*

$$\log^* n = \min\left\{k \in \mathbb{N} : \log^{(k)} n \leq 1\right\} \tag{6}$$

for positive integer n. (i): Compute $\log^{(2)} 16, \log^{(3)} 256, \log^{(3)} 2^{65536}$. *(ii): Compute* $\log^* 2, \log^* 4, \log^* 2^{2048}$.

Proof. (i): $\log^{(2)} 16 = \log^{(1)}(\log_2 16) = \log^{(1)} 4 = \log^{(0)}(\log_2 4) = \log^{(0)} 2 = 2$.

$\log^{(3)} 256 = \log^{(2)}(\log_2 256) = \log^{(2)} 8 = \log^{(1)}(\log_2 8) = \log^{(1)} 3 = \log^{(0)}(\log_2 3) = \log_2 3$.

$\log^{(3)} 2^{65536} = \log^{(2)}(\log_2 2^{65536}) = \log^{(2)} 65536 = \log^{(1)}(\log_2 65536) = \log^{(1)} 16 = \log^{(0)}(\log_2 16) = \log^{(0)} 4 = 4$.

(ii): $\log^{(0)} 2 = 2 > 1, \log^{(1)} 2 = \log_2 2 = 1 \leq 1$, so $\log^* 2 = 1$.

$\log^{(0)} 4 = 4 > 1, \log^{(1)} 4 = \log_2 4 = 2 > 1, \log^{(2)} 4 = \log_2 \log^{(1)} 4 = \log_2 2 = 1 \leq 1$, so $\log^* 4 = 2$.

$\log^{(0)} 2^{2048} = 2^{2048} > 1, \log^{(1)} 2^{2048} = \log_2 2^{2048} = 2048 > 1, \log^{(2)} 2^{2048} = \log_2 \log^{(1)} 2^{2048} = \log_2 2048 = 11 > 1, \log^{(3)} 2^{2048} = \log_2 \log^{(2)} 2^{2048} = \log_2 11 > 1$ since $11 > 2^1 = 2$.

Now $\log^{(4)} 2^{2048} = \log_2 \log^{(3)} 2^{2048} = \log_2 \log_2 11 > 1$ since $\log_2 11 > 2^1 = 2$ this is because $11 > 2^2 = 4$.

Continue the iteration $\log^{(5)} 2^{2048} = \log_2 \log^{(4)} 2^{2048} = \log_2 \log_2 \log_2 11 \leq 1$ since $\log_2 \log_2 11 \leq 2^1 = 2$ this is because $\log_2 11 \leq 2^2 = 4$ this is because $11 \leq 2^4 = 16$.

As a result, $\log^* 2^{2048} = 5$.

$\square$

Remark. *We can see that* $\log^* n = k$ *if and only if* $2^{2^{2\cdots}} < n \leq 2^{2^{2\cdots}}$ *where there are* $k-1$ *copies of '2' appearing on LHS and* k *copies of '2' appearing on RHS. Since* $2^{2^{2^2}} = 2^{2^4} < 2^{2048} = 2^{2^{11}} < 2^{2^{16}} = 2^{2^{2^{2^2}}}$*, this is consistent with our calculation result that* $\log^* 2^{2048} = 5$*.*