BM $\rightarrow$ finite-dim dist

BM $\rightarrow$ sample path continuity

Path regularity is crucial in cts-time setting.

e.g: $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$, $\mathbb{P} = \lambda$,

$X_t(\omega) = I_{\{t=\omega\}}$, $Y_t(\omega) = 0$ for $t \in [0,1]$.

In this case, $\forall t \in [0,1]$, $Y_t(\omega)$ is constantly zero but $X_t(\omega)$ is only non-zero at $\omega = t$, then $X_t = Y_t$ a.s. since any single real number has zero Lebesgue measure.

However, $\{Y_t\}$ always has cts sample path while $\{X_t\}$ does not, $\mathbb{P}(\sup_{t\in[0,1]} X_t = 0) = 0$, $\mathbb{P}(\sup_{t\in[0,1]} Y_t = 0) = 1$.

Fix ω:

$\{X_t\}$ is a modification of $\{Y_t\}$ if

$\forall t$, $X_t = Y_t$ a.s.

$\{X_t\}$ is indistinguishable from $\{Y_t\}$ if $\exists \boxed{N}$,

$\mathbb{P}(N)=0$, $\forall \omega \notin N$, $\forall t$, $X_t(\omega) = Y_t(\omega)$

uniform w.r.t. time

In the example above, they are not indistinguishable.

e.g.: $\{X_t : t\in[0,1]\}$ i.i.d. $N(0,1)$, show that it can't have a.s. cts sample path.

Pf: $\forall \varepsilon>0$, $\mathbb{P}(X_t>\varepsilon, X_{t+\frac{1}{n}}<-\varepsilon)$

$$= \mathbb{P}(X_t>\varepsilon)\cdot \mathbb{P}(X_{t+\frac{1}{n}}<-\varepsilon)$$

$$= [\Phi(-\varepsilon)]^2 \not\to 0 \quad (n\to\infty)$$

which means there's always positive prob that X_t and $X_{t+\frac{1}{n}}$ are far enough even for large n.

Back to BM: sample path cts is from

Kolmogorov's lemma (sample path α-Hölder cts for $\forall \alpha \in (0, \frac{1}{2})$)

Check: $\{W_t^2 - t\}$ is MG, $\{e^{\lambda W_t - \frac{1}{2}\lambda^2 t}\}$ is MG

$\downarrow$ quad $\qquad\qquad$ $\downarrow$ exp.

Can they be generalized? Actually,

$$\begin{cases} W_t^2 - t = W_t^2 - \underbrace{\langle W, W\rangle_t}_{\text{quad var until time } t} & \Longrightarrow \text{Doob's decomposition} \\ e^{\lambda W_t - \frac{1}{2}\lambda^2 t} = e^{\lambda W_t - \frac{1}{2}\underbrace{\langle \lambda W, \lambda W\rangle_t}_{\text{quad}}} & \Downarrow \end{cases}$$

Stochastic exponential $\mathcal{E}(\lambda W)_t$

2. Let G be a standard normal random variable and $(W_t,\ 0 \le t < \infty)$ another standard Brownian motion. Assume that G, (B_t) and (W_t) are independent and then define the process $(Y_t)_{t\ge 0}$ by

$$Y_t := \begin{cases} B_t\,, & 0 \le t \le 1\,, \\ \sqrt{t}\,(B_1 \cos(W_{\log t}) + G \sin(W_{\log t}))\,, & t \ge 1\,. \end{cases}$$

(a) Compute the marginal distribution of Y_t for every $t \ge 0$ (*Hint:* for $t \ge 1$ compute the characteristic function of Y_t by conditioning on $W_{\log t}$).

(b) Explain why $(Y_t,\ t \ge 0)$ is a continuous martingale with respect to its own filtration:

$$\mathcal{G}_t := \begin{cases} \sigma\{B_s, 0 \le s \le t\}\,, & 0 \le t \le 1\,, \\ \sigma\{B_1, G, (W_s, 0 \le s \le \log t)\}\,, & t > 1\,. \end{cases}$$

You do not have to be fully rigorous in this part, and may rely on the fact that one can generalize the exponential martingales to the complex plane $\mathbb{C}$: for any real constant $c \in \mathbb{R}$, $(e^{icB_t + \frac{c^2}{2}t}, 0 \le t < \infty)$ is a complex-valued martingale .

(c) Show that despite parts (a)-(b) the process $(Y_t,\ t \ge 0)$ is NOT a Brownian motion! (*Hint:* show that $Y_e - Y_1$ is not Gaussian). This is an example of a *fake* Brownian Motion.

(a): $\forall\ 0 \le t \le 1,\quad Y_t = B_t \sim N(0,t)$

$\forall\ t \ge 1,\quad Y_t = \sqrt{t}\left[B_1 \cdot \cos(W_{\log t}) + G \cdot \sin(W_{\log t})\right]$

$\phi_{Y_t}(s) = \mathbb{E}e^{isY_t} = \mathbb{E}\left[\mathbb{E}\left(e^{isY_t} \mid W_{\log t}\right)\right]$

$\mathbb{E}\left(e^{isY_t} \mid W_{\log t} = k\right) = \mathbb{E}\left(e^{is\sqrt{t}(B_1 \cdot \cos k + G \cdot \sin k)} \mid W_{\log t} = k\right)$

indep

$= \mathbb{E}\, e^{is\sqrt{t}(B_1 \cos k + G \sin k)}$

$= \underbrace{\mathbb{E}e^{is\sqrt{t}B_1 \cos k} \cdot \mathbb{E}e^{is\sqrt{t} \cdot G \sin k}}_{\text{indep}}$

$= \phi_{B_1}(s\sqrt{t}\cos k) \cdot \phi_G(s\sqrt{t}\sin k)$,

$$\boxed{\begin{aligned}\phi_{B_1} &= \phi_G \\ &= e^{-\frac{1}{2}t^2}\end{aligned}}$$

$= e^{-\frac{1}{2}s^2 t \cos^2 k} \cdot e^{-\frac{1}{2}s^2 t \sin^2 k}$

$= e^{-\frac{1}{2}s^2 t}$

So $\phi_{Y_t}(s) = \mathbb{E}\left[e^{-\frac{1}{2}s^2 \cdot t}\right] = e^{-\frac{1}{2}s^2 t}$

so $\underline{Y_t \sim N(0, \sqrt{t})}$.

(b): $\forall t \geq s \geq 0$, when t, s both ≤ 1, obvious.

When $t > 1 \geq s$, $\mathbb{E}(Y_t | \mathcal{F}_s) = \sqrt{t} \cdot \Big(\mathbb{E}[B_1 \cos(W_{\log t}) | \mathcal{F}_s]$

G & W indep of $\mathcal{F}_s$

$+ \underbrace{\mathbb{E}[G \sin(W_{\log t})]}_{\mathbb{E}G = 0,\ = 0.} \Big)$

$= \sqrt{t} \cdot \Big[\mathbb{E}\ \underbrace{(B_1 - B_s)\cos(W_{\log t})}_{\text{indep of } \mathcal{F}_s} + \mathbb{E}[B_s \cos(W_{\log t}) | \mathcal{F}_s] \Big)$

$= \sqrt{t} \cdot \Big(\underbrace{\mathbb{E}(B_1 - B_s)}_{=0} \cdot \mathbb{E}\cos(W_{\log t}) + B_s \cdot \mathbb{E}[\cos(W_{\log t})] \Big)$

$= \sqrt{t} \cdot B_s \cdot \underbrace{\mathbb{E}\cos(W_{\log t})}_{W_{\log t} \sim N(0, \log t)}$ $\qquad e^{is W_{\log t}}$

Now $\phi_{W_{\log t}}(s) = e^{-\frac{1}{2}(\log t) \cdot s^2}$, plug in $s = 1$

$\mathbb{E} e^{i \cdot W_{\log t}} = e^{-\frac{1}{2}\log t}$, take real parts,

$\mathbb{E}\cos(W_{\log t}) = t^{-\frac{1}{2}}$, so $\mathbb{E}(Y_t | \mathcal{F}_s) = B_s$

$= Y_s$ ✓

When $\forall t \geq s > 1$,

$$\mathbb{E}(Y_t | \mathcal{F}_s) = \sqrt{t} \cdot \left(\left[\underbrace{B_1}_{\text{meas.}} \cdot \mathbb{E}\left[\cos(W_{\log t}) \mid \mathcal{F}_s\right] + \underbrace{G}_{\text{meas.}} \cdot \mathbb{E}\left[\sin(W_{\log t}) \mid \mathcal{F}_s \right] \right) \right.$$

here $\mathbb{E}\left[e^{i \cdot W_{\log t}} \mid \mathcal{F}_s\right]$

$$= e^{\frac{1}{2}\log s - \frac{1}{2}\log t} \cdot e^{i \cdot W_{\log s}}$$

$$= \sqrt{\frac{s}{t}} \cdot e^{i \cdot W_{\log s}}$$

(since $e^{i W_{\log t} + \frac{1}{2}\log t}$ is MG)

so: $\mathbb{E}\left[\cos(W_{\log t}) \mid \mathcal{F}_s\right] = \sqrt{\frac{s}{t}} \cdot \cos(W_{\log s})$

$\mathbb{E}\left[\sin(\text{——}) \mid \text{—}\right] = \sqrt{\frac{s}{t}} \cdot \sin(\text{——})$

So: $\mathbb{E}(Y_t | \mathcal{F}_s) = Y_s$ ✓

(c): $Y_e - Y_1 = \sqrt{e}\cdot\left[B_1\cdot\cos(W_1) + G\cdot\sin(W_1)\right]$

$- B_1$

$= \left[\sqrt{e}\cos(W_1) - 1\right] B_1 + \sqrt{e}\cdot G\cdot\sin(W_1)$

$$\phi_{Y_e - Y_1}(s) = \mathbb{E}\left[\mathbb{E}\left(e^{is(Y_e - Y_1)} \mid W_1\right)\right]$$

$\mathbb{E}(\text{--} \mid W_1 = k)$

$$= \mathbb{E}\left(e^{is(\sqrt{e}\cos k - 1)\cdot B_1}\cdot e^{is\sqrt{e}G\cdot\sin k} \mid W_1 = k\right)$$

indep

$$= \mathbb{E}\, e^{is(\sqrt{e}\cos k - 1)\cdot B_1}\cdot \mathbb{E}\, e^{is\sqrt{e}\, G\cdot\sin k}$$

$$= e^{-\frac{1}{2}s^2(\sqrt{e}\cos k - 1)^2}\cdot e^{-\frac{1}{2}(\sqrt{e}\sin k)^2\cdot s^2}$$

$$= e^{-\frac{1}{2}s^2\cdot\left[(\sqrt{e}\cos k - 1)^2 + (\sqrt{e}\sin k)^2\right]}$$

$$= e^{-\frac{1}{2}s^2\cdot(e + 1 - 2\sqrt{e}\cos k)}$$

So: $\phi_{Y_e - Y_1}(s) = \mathbb{E}\left[e^{-\frac{1}{2}s^2\cdot(e+1-2\sqrt{e}\cos W_1)}\right]$

$$= e^{-\frac{1}{2}s^2\cdot(e+1)} \cdot \mathbb{E}\left[e^{s^2\sqrt{e}\cos W_1}\right]$$

Assume $Y_e - Y_1$ is Gaussian, then

$$\exists\ \sigma^2 \geq 0, \quad \mathbb{E}\, e^{s^2\sqrt{e}\cos W_1} = e^{-\frac{1}{2}\sigma^2 s^2}$$

if this is true,

$$\mathbb{E}\, e^{4\sqrt{e}\cos W_1} = \mathbb{E}\left(e^{\sqrt{e}\cos W_1}\right)^4$$

(s=2) $\|$ $\quad$ $\perp$ (Jensen, strict!)

$$e^{-2\sigma^2} \qquad \left(\mathbb{E}\, e^{\sqrt{e}\cos W_1}\right)^4$$

$\|$ (s=1)

$$\left(e^{-\frac{1}{2}\sigma^2}\right)^4 = e^{-2\sigma^2}$$

Contradiction!

So: from c.f., $Y_e - Y_1$ not Gaussian,

(Y_t) not BM!