OST

- condition
  - T a.s. bounded
    (a.s. finite not suffice!)
  - or
  - U.I. MG
  - or
  - a.s. uniformly bounded increment + $T \in \mathcal{L}^1$
    (useful for random walk typically)
  - or
  - apply for $T \wedge n$
    (and set $n \to \infty$, use convergence thms)
  - $\mathbb{E} Y_T \le \mathbb{E} Y_0$
    (non-neg super-MG)
- application
  - $\mathbb{P}(T_1 < T_2)$
    (OST for $\{X_n\}$ itself)
  - $\mathbb{E} T$
    (OST for quadratic MG, e.g., $W_t^2 - t$)
  - Law(T)
    (OST for exponential MG, e.g., $e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$)

12.5.1: $\{Y_n\}$ MG, T stp time, $T<\infty$ a.s.

show that $\mathbb{E}Y_T = \mathbb{E}Y_0$ if either one of following holds:

(a): $\mathbb{E}\sup_n |Y_{T\wedge n}| < \infty$

(b): $\exists\, c, \delta > 0, \forall n, \mathbb{E}|Y_{T\wedge n}|^{1+\delta} \le C$

Pf:

If (a) holds: apply OST for $T\wedge n$ to get $\mathbb{E}Y_{T\wedge n} = \mathbb{E}Y_0$ for $\forall n$.

Set $n\to\infty$, $T\wedge n \xrightarrow{a.s.} T$, we hope to see

$$\underline{Y_{T\wedge n} \xrightarrow{a.s.} Y_T}$$

$$\Downarrow$$

check: $|Y_{T\wedge n} - Y_T| = |Y_n|\cdot I_{\{T>n\}} \xrightarrow{a.s.} 0 \quad (n\to\infty)$

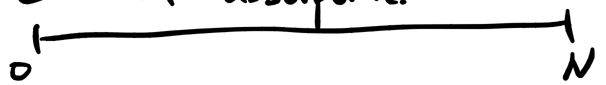
since $T<\infty$ a.s.

(a) holds $\xRightarrow{DCT}$ $\mathbb{E}Y_{T\wedge n} \to \mathbb{E}Y_T \ (n\to\infty)$,

proves $\mathbb{E}Y_T = \mathbb{E}Y_0$.

If (b) holds: $\{Y_{T\wedge n}\}$ u.I., OST holds for u.I. MG that $\mathbb{E}Y_T = \mathbb{E}Y_{T\wedge 0} = \mathbb{E}Y_0$.

12.5.4: $\{S_n\}$ SRW, $0 < S_0 < N$ a.s., absorbing barriers at 0 and N. Compute prob of absorption at 0 and mean time until absorption.



$$\underbrace{T_0 \qquad\qquad T_N}_{\text{first hitting time}}$$

absorption time $T = T_0 \wedge T_N$

$\{S_n\}$ is MG and $\forall n$, $0 \le S_{n \wedge T} \le N$ a.s.

so $\{S_{n\wedge T}\}$ is a.s. bounded $\Rightarrow$ u.I.

By OST, $\mathbb{E} S_T = \mathbb{E} S_0$

$$\mathbb{E} S_T = \underbrace{\mathbb{P}(T_0 < T_N) \cdot 0 + \mathbb{P}(T_0 > T_N) \cdot N}_{\text{add up to 1}}$$

so: $\begin{cases} \mathbb{P}(T_0 < T_N) = 1 - \frac{1}{N} \mathbb{E} S_0 \\ \mathbb{P}(T_0 > T_N) = \frac{1}{N} \mathbb{E} S_0 \end{cases}$

Now want to compute $\mathbb{E}T$, use quad MG

$$Y_n = S_n^2 - n, \quad \mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}\left((S_n+\xi_{n+1})^2 \,\middle|\, \mathcal{F}_n\right) - (n+1)$$

$$= S_n^2 + 2S_n\,\mathbb{E}\xi_{n+1} + \mathbb{E}\xi_{n+1}^2 - (n+1)$$

$$= S_n^2 - n = Y_n \quad \checkmark$$

Apply OST for $T\wedge n$:

$$\mathbb{E}Y_{T\wedge n} = \mathbb{E}Y_0 = \mathbb{E}S_0^2$$

$$\|$$

$$\mathbb{E}S_{T\wedge n}^2 - \mathbb{E}(T\wedge n)$$

$T\wedge n \overset{a.s.}{\nearrow} T$, by MCT, $\mathbb{E}(T\wedge n) \xrightarrow{a.s.} \mathbb{E}T$ $(n\to\infty)$

previous problem tells the dist!

$S_{T\wedge n}^2 \xrightarrow{a.s.} S_T^2$ is either 0^2 or N^2 a.s.

and $|S_{T\wedge n}^2| \le N^2$ a.s., by BCT, $\mathbb{E}S_{T\wedge n}^2 \to \mathbb{E}S_T^2$ $(n\to\infty)$

So: $\mathbb{E}S_0^2 = \left(1-\frac{1}{N}\mathbb{E}S_0\right)\cdot 0^2 + \frac{1}{N}\mathbb{E}S_0\cdot N^2 - \mathbb{E}T$

$$\underline{\mathbb{E}T = \mathbb{E}\left[S_0(N-S_0)\right]}$$

12.5.5 : $\{S_n\}$ SRW with $S_0 = 0$, show that

$$Y_n = \frac{\cos\left(\lambda\left(S_n - \frac{b-a}{2}\right)\right)}{\cos^n \lambda}$$ is MG if $\cos\lambda \neq 0$.

Let a, b be positive int, show that if T is the time till absorption on $[-a, b]$, then

$$\mathbb{E}(\cos\lambda)^{-T} = \frac{\cos \frac{\lambda(b-a)}{2}}{\cos \frac{\lambda(b+a)}{2}} \quad \left(0 < \lambda < \frac{\pi}{b+a}\right)$$

Pf: Y_n adapted, $\mathbb{E}|Y_n| < \infty$,

$$\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) = \cos^{-n-1}\lambda \cdot \mathbb{E}\left[\cos\left(\lambda\left(\overset{S_n + \xi_{n+1}}{S_{n+1}} - \frac{b-a}{2}\right)\right) \Big| \mathcal{F}_n\right]$$

$$= \cos^{-n-1}\lambda \cdot \Big\{\mathbb{E}\Big(\cos\left(\lambda\left(S_n - \tfrac{b-a}{2}\right)\right) \cdot \cos(\lambda \xi_{n+1})$$
$$- \sin\left(\lambda\left(S_n - \tfrac{b-a}{2}\right)\right) \cdot \sin(\lambda \xi_{n+1}) \Big| \mathcal{F}_n\Big)\Big\}$$

$$= \cos^{-n-1}\lambda \cdot \Big[\cos \lambda\left(S_n - \tfrac{b-a}{2}\right) \cdot \underbrace{\mathbb{E}\cos(\lambda \xi_{n+1})}_{\cos\lambda}$$
$$- \sin \lambda \cdot \left(S_n - \tfrac{b-a}{2}\right) \cdot \underbrace{\mathbb{E}\sin(\lambda \xi_{n+1})}_{= 0}\Big]$$

$$= \cos^{-n}\lambda \cdot \cos \lambda\left(S_n - \tfrac{b-a}{2}\right)$$

$$= Y_n \quad \checkmark$$

Before OST:

$$\frac{\cos\frac{\lambda(a+b)}{2}}{(\cos\lambda)^{T\wedge n}} \leq Y_{T\wedge n} \leq \frac{1}{(\cos\lambda)^{T}} \quad \text{a.s.}$$

for $\forall n$.

(since $-a \leq S_{T\wedge n} \leq b$)

prove: $\mathbb{E}\frac{1}{(\cos\lambda)^T} < \infty$!

Since $\{Y_n\}$ non-neg, $\mathbb{E}Y_{T\wedge n} \leq \mathbb{E}Y_0$ by OST,

so $\mathbb{E}Y_0 \geq \cos\frac{\lambda(a+b)}{2} \cdot \mathbb{E}\frac{1}{(\cos\lambda)^{T\wedge n}}$

By Fatou, $\mathbb{E}\frac{1}{(\cos\lambda)^T} \leq \varliminf_{n\to\infty} \mathbb{E}\frac{1}{(\cos\lambda)^{T\wedge n}}$

$$\leq \frac{\mathbb{E}Y_0}{\cos\frac{\lambda(a+b)}{2}} < \infty$$

So: $Y_{T\wedge n}$ is dominated by some $\mathcal{L}^1$ r.v., $\{Y_{T\wedge n}\}$ is U.I., apply OST,

$$\underline{\underline{\mathbb{E}(\cos\lambda)^{-T}}} \cdot \cos\frac{\lambda(a+b)}{2} = \mathbb{E}Y_T = \mathbb{E}Y_0 = \cos\frac{\lambda(b-a)}{2}$$

$$\mathbb{E}(\cos\lambda)^{-T} = \mathbb{E}\left(\boxed{\frac{1}{\cos\lambda}}\right)^{T}$$

$$\| $$

$$\frac{\cos\frac{\lambda(a-b)}{2}}{\cos\frac{\lambda(a+b)}{2}}$$

$$\cos\lambda = \frac{e^{i\lambda}+e^{-i\lambda}}{2}$$

wisdom of

leaving the unspecified parameter λ in the MG!

e.g.: Calculate $\mathbb{E}T$,

$$\frac{d}{d\lambda}\mathbb{E}(\cos\lambda)^{-T} = \sin\lambda\cdot\mathbb{E}\left[T\cdot(\cos\lambda)^{-T-1}\right]$$

$$\|$$

$$\frac{\frac{a+b}{2}\cdot\sin\frac{\lambda(a+b)}{2}\cdot\cos\frac{\lambda(a-b)}{2} - \frac{a-b}{2}\cdot\sin\frac{\lambda(a-b)}{2}\cdot\cos\frac{\lambda(a+b)}{2}}{\cos^2\frac{\lambda(a+b)}{2}}$$

Set $\lambda=0$: $\mathbb{E}T = \lim\limits_{\lambda\to0^+}\dfrac{\frac{a+b}{2}\sin\frac{\lambda(a+b)}{2} - \frac{a-b}{2}\sin\frac{\lambda(a-b)}{2}}{\sin\lambda}$

$$= \boxed{ab}$$

matches the previous problem ($S_0=a$, $N=b+a$),

$\mathbb{E}T = \mathbb{E}\,S_0(N-S_0) = ab$ ✓