12.4.1: T_1, T_2 are stp times w.r.t. $\{\mathcal{G}_n\}$, show that T_1+T_2, $\max\{T_1, T_2\}$, $\min\{T_1, T_2\}$ are stp times.

Def of stopping time: $\forall n$, $\underbrace{\{T \le n\} \in \mathcal{G}_n}$ or $\forall n, \{T=n\} \in \mathcal{G}_n$

$\Downarrow$

one can determine if the stopping criterion is met at time n based on all information up to time n.

Pf:

$$\forall n, \{T_1+T_2=n\} = \bigcup_{k=0}^{n} \{T_1=k, T_2=n-k\}$$

since $\{T_1=k\} \in \mathcal{G}_k \subseteq \mathcal{G}_n$, $\{T_2=n-k\} \in \mathcal{G}_{n-k} \subseteq \mathcal{G}_n$,

$\{T_1=k, T_2=n-k\} \in \mathcal{G}_n$ for $\forall k \in \{0, 1, \cdots, n\}$ so the union is still in $\mathcal{G}_n$. ✓

$\forall n, \{\max\{T_1, T_2\} \le n\} = \{T_1 \le n, T_2 \le n\} \in \mathcal{G}_n$ ✓

$\forall n, \{\min\{T_1, T_2\} > n\} = \{T_1 > n, T_2 > n\}$

$\{T_1 > n\} = \{T_1 \le n\}^c \in \mathcal{G}_n$, $\{T_2 > n\} = \{T_2 \le n\}^c \in \mathcal{G}_n$

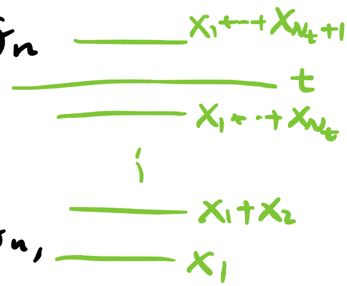
so $\{T_1 > n, T_2 > n\} \in \mathcal{G}_n$ ✓

12.4.2: $X_1, X_2, \cdots$ be non-neg independent and $N_t = \max\{n : X_1 + \cdots + X_n \le t\}$. Show that $N_t + 1$ is a stopping time w.r.t. suitable filtration.

Pf: Want to find $\{\mathcal{G}_n\}$ s.t.

$$\forall n, \ \{N_t + 1 \le n\} = \{N_t \le n-1\} \in \mathcal{G}_n$$

Now $\{N_t \le n-1\} = \{X_1 + \cdots + X_n > t\}$



To ensure that $\forall n, \{X_1 + \cdots + X_n > t\} \in \mathcal{G}_n$, naturally specify $\mathcal{G}_n = \sigma(X_1, \cdots, X_n)$.

12.4.7: For any stp time S, T w.r.t. $\{\mathcal{G}_n\}$,

$$\mathcal{G}_T \triangleq \{A : \forall n, A \cap \{T \le n\} \in \mathcal{G}_n\}.$$ $\Rightarrow$ all information until process stopped by T

(a): Show that $\mathcal{G}_T$ is a σ-field, and $T \in \mathcal{G}_T$.

(b): If $A \in \mathcal{G}_S$, then $A \cap \{S \le T\} \in \mathcal{G}_T$

(c): If $S \le T$, then $\mathcal{G}_S \subseteq \mathcal{G}_T$.

Pf:

(a): $\phi, \Omega \in \mathcal{G}_T$ ✓

$\forall A, B \in \mathcal{G}_T$, $A \subseteq B$, then $\forall n$, $(B - A) \cap \{T \le n\}$

$= B \cap \{T \le n\} - A \cap \{T \le n\} \in \mathcal{G}_n$, so $B - A \in \mathcal{G}_T$ ✓

$\forall A_1, A_2, \cdots \in \mathcal{G}_T$, then $\forall n$, $\left(\bigcup_{k=1}^{\infty} A_k\right) \cap \{T \le n\}$

$= \bigcup_{k=1}^{\infty} \left(A_k \cap \{T \le n\}\right) \in \mathcal{G}_n$, so $\bigcup_{k=1}^{\infty} A_k \in \mathcal{G}_T$ ✓

Since T is integer-valued, and $\forall n$, $\{T \le n\} \in \mathcal{G}_n$, it implies $T \in \mathcal{G}_T$.

(b): If $A \in \mathcal{G}_S$, $\forall n$, $A \cap \{S \le T\} \cap \{T \le n\}$

$$= A \cap \bigcup_{k=0}^{n} \{T=k, S \le k\} = \bigcup_{k=0}^{n} \underbrace{\left(\underbrace{A \cap \{S \le k\}}_{\in \mathcal{G}_k} \cap \underbrace{\{T=k\}}_{\in \mathcal{G}_k}\right)}_{\in \mathcal{G}_n}$$

$\in \mathcal{G}_n$, so $A \cap \{S \le T\} \in \mathcal{G}_T$.

(c): If $S \le T$, then $\forall A \in \mathcal{G}_S$,

consider $\forall n$, $A \cap \{T \le n\} = A \cap \left(\bigcup_{k=0}^{n} \{T=k\}\right)$

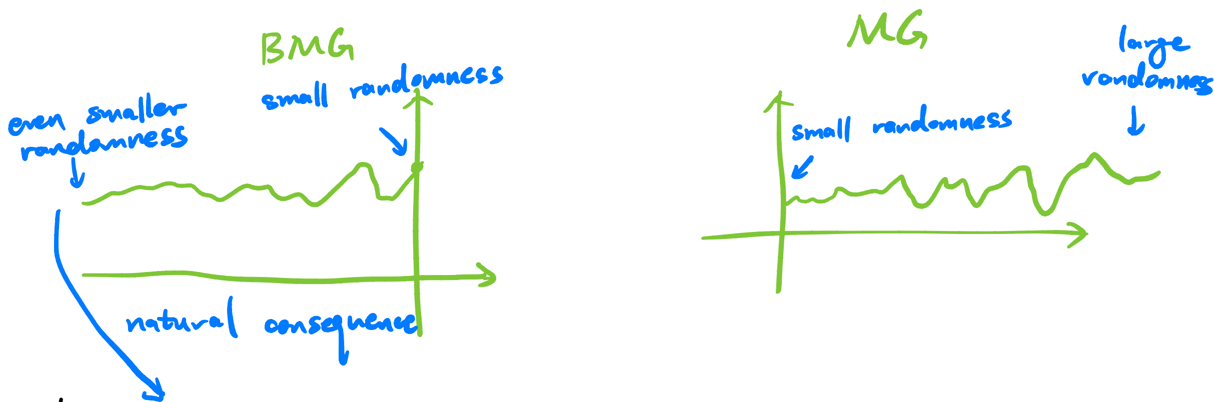
$$= \bigcup_{k=0}^{n} \left(A \cap \{T=k\}\right) = \bigcup_{k=0}^{n} \underbrace{\left(\underbrace{A \cap \{S \le k\}}_{\in \mathcal{G}_k} \cap \underbrace{\{T=k\}}_{\in \mathcal{G}_k}\right)}_{\in \mathcal{G}_n} \in \mathcal{G}_n$$

so $A \in \mathcal{G}_T$. Then $\mathcal{G}_S \in \mathcal{G}_T$.

Backward MG: $\{X_n\}$ $\mathcal{L}^1$, adapted to $\{\mathcal{G}_n\}$, $n \le 0$,

$\forall n \le -1$, $\mathbb{E}(X_{n+1} | \mathcal{G}_n) = X_n$.

Difference is in filtration, now the largest σ-field in the filtration is $\mathcal{G}_0$, unlike normal MG when the largest σ-field is at time ∞.



Thm: $X_n \xrightarrow[\mathcal{L}^1]{a.s.} X_{-\infty}$ $(n \to -\infty)$ for any backward MG.

Pf: $U_n^{a,b}$ = # of upcrossing of $[a,b]$ by $X_{-n}, \ldots, X_0$

then by Doob's upcrossing inequality, $\mathbb{E} U_n^{a,b} \le \frac{\mathbb{E}(X_0 - a)^+}{b-a}$

set $n \to \infty$, by MCT, $\mathbb{E} U_\infty^{a,b} \le \frac{\mathbb{E}(X_0 - a)^+}{b-a} < \infty$ for $\forall a < b$ implies that $X_n \xrightarrow{a.s.} X_{-\infty}$ $(n \to -\infty)$.

$\mathcal{L}^1$ convergence is from the fact that $X_n = \mathbb{E}(X_0 | \mathcal{G}_n)$ for $\forall n \le -1$, it's a closed MG.

Identify the limit $X_{-\infty}$?

$\mathcal{G}_{-\infty} = \bigcap_{n \le 0} \mathcal{G}_n$, then $X_{-\infty} = \mathbb{E}(X_0 | \mathcal{G}_{-\infty})$ from the structure as a closed MG.

e.g.: $\xi_1, \xi_2, \cdots$ i.i.d., $\mathcal{L}^1$, $S_n = \xi_1 + \cdots + \xi_n$, $X_{-n} = \frac{S_n}{n}$,

then $\{X_{-n}\}$ $\mathcal{L}^1$, adapted to $\mathcal{G}_{-n} = \sigma(S_n, \xi_{n+1}, \xi_{n+2}, \cdots)$.

Check: $\mathcal{G}_{-n} = \sigma(S_n, \xi_{n+1}, \xi_{n+2}, \cdots) \subseteq \underbrace{\sigma(S_{n-1}, \xi_n, \xi_{n+1}, \cdots)}_{= \mathcal{G}_{-n+1}}$

so $\{\mathcal{G}_{-n}\}$ is a filtration.

Now $\{X_{-n}\}$ $\mathcal{L}^1$ and adapted to $\{\mathcal{G}_{-n}\}$, with

$$\mathbb{E}(X_{-n+1} | \mathcal{G}_{-n}) = \mathbb{E}\left(\frac{S_{n-1}}{n-1} \Big| S_n, \xi_{n+1}, \xi_{n+2}, \cdots\right)$$

$$= \frac{1}{n-1}\left[S_n - \mathbb{E}(\xi_n | S_n, \xi_{n+1}, \cdots)\right]$$

$$= \frac{1}{n-1}\left[S_n - \underline{\mathbb{E}(\xi_n | S_n)}\right]$$

symmetry

$$= \frac{1}{n-1}\left(S_n - \underline{\frac{1}{n} S_n}\right) = \frac{S_n}{n} = X_{-n}$$

so $\{X_{-n}\}$ is BMG.

The convergence thm implies $X_{-n} \xrightarrow[\mathcal{L}^1]{a.s.} X_{-\infty}$

with $X_{-\infty} = \mathbb{E}(X_{-1} | \mathcal{G}_{-\infty}) = \mathbb{E}(\xi_1 | \mathcal{G}_{-\infty})$.

Clearly $\mathcal{F}_{-n} \subseteq \underline{\mathcal{E}_n}$, since $\mathcal{F}_{-n} = \sigma(S_n, \xi_{n+1}, \cdots)$

σ-field invariant under any permutation in first n components of sample point

S_n does not change if the value of $\xi_1, \cdots, \xi_n$ is permuted since it's always the sum, and $\xi_{n+1}, \xi_{n+2}, \cdots$ are not affected by the permutation.

Now that $\mathcal{F}_{-\infty} = \bigcap_{n=1}^{\infty} \mathcal{F}_{-n} \subseteq \underline{\bigcap_{n=1}^{\infty} \mathcal{E}_n = \mathcal{E}}$,

exchangeable σ-field, invariant under any finite permutation

due to Hewitt-Savage 0-1 law, i.i.d. r.v. has trivial exchangeable σ-field, so $\mathcal{F}_{-\infty}$ is trivial.

So $X_{-\infty} = \mathbb{E}\xi_1$ and we have proved

$$\frac{S_n}{n} \xrightarrow[\mathcal{L}^1]{a.s.} \mathbb{E}\xi_1 \quad (n \to \infty),$$

which is SLLN!

SLLN is just the consequence of MG convergence for backward MG!