12.3.3: $\{X_n\}$ is Markov chain with countable state space S, transition matrix P. If $\{X_n\}$ is irreducible and recurrent, and $\exists\, \psi : S \to S$ bounded s.t. $\sum_{j \in S} P_{ij}\, \psi(j) \le \psi(i)$ for $\forall\, i \in S$, then ψ is constant.

super-harmonic

Pf:

From what we have proved, $\{\psi(X_n)\}$ is super-MG and it's a.s. bounded, so $\psi(X_n)$ converges a.s. by MG convergence thm.

However, $\{X_n\}$ recurrent so $\{\psi(X_n)\}$ is recurrent. If ψ is not constant, i.e. range(ψ) contains $a, b \in S$, $a \neq b$, since $\{\psi(X_n)\}$ irreducible, starting from $\psi(X_0) = a$, $\{\psi(X_n)\}$ visits a and b infinitely often a.s., contradiction with a.s. convergence!

Analogue to: super-harmonic functions on $\mathbb{R}^2$ bounded from below is constant.

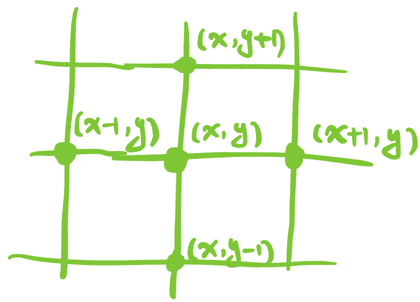
Another e.g. of using MG techniques to prove analysis argument:

Consider $f: \mathbb{Z}^2 \to \mathbb{R}$ bounded, satisfying $\forall (x,y) \in \mathbb{Z}^2$,

$$f(x,y) = \frac{f(x,y+1) + f(x,y-1) + f(x-1,y) + f(x+1,y)}{4},$$ then

such f must be constant.

interpretation:



function value = local average of function values on neighboring grid points

Pf:

Consider $\{S_n\}$ as a SRW on $\mathbb{Z}^2$, then $\{f(S_n)\}$ is a MG, since it's non-neg, it converges a.s.

Since $\{S_n\}$ is recurrent, $\{f(S_n)\}$ is also recurrent, resulting in f constant.

The result does not necessarily hold on $\mathbb{Z}^d$ $(d \geq 3)$ since SRW is transient on $\mathbb{Z}^d$ $(d \geq 3)$.

12.3.4: $Z_1, Z_2, \ldots$ independent s.t.

$$Z_n = \begin{cases} a_n & \text{w.p. } \frac{1}{2n^2} \\ 0 & \text{w.p. } 1-\frac{1}{n^2} \\ -a_n & \text{w.p. } \frac{1}{2n^2} \end{cases}$$

where $a_1 = 2$, $a_n = 4\sum_{j=1}^{n-1} a_j$. Show that $Y_n = \sum_{j=1}^{n} Z_j$ is a MG. Show that the limit of Y_n exists a.s. but there's no M s.t. $\sup_n \mathbb{E}|Y_n| \le M$.

$a_{n-1} = 4(a_1 + \cdots + a_{n-2})$

Pf: $\{Y_n\}$ adapted to $\mathcal{G}_n \triangleq \sigma(Z_1, \ldots, Z_n)$, $\forall n$, $\mathbb{E}|Y_n| \le \sum_{j=1}^{n} \mathbb{E}|Z_j| = \sum_{j=1}^{n} \frac{a_j}{j^2} < \infty$.

$$\mathbb{E}(Y_{n+1}|\mathcal{G}_n) = Y_n + \mathbb{E}(Z_{n+1}|\mathcal{G}_n) = Y_n + \mathbb{E}Z_{n+1} = Y_n$$

So $\{Y_n\}$ is a MG.

By Borel-Cantelli; since $\mathbb{P}(Z_n \ne 0) = \frac{1}{n^2}$, $\sum_n \mathbb{P}(Z_n \ne 0) < \infty$ so $\mathbb{P}(Z_n \ne 0 \text{ i.o.}) = 0$, this implies $\mathbb{P}(Z_n = 0 \text{ eventually}) = 1$ so Y_n converges a.s. to some limit.

However, $a_n = 4(a_1 + \cdots + a_{n-1}) = 4\left(\frac{a_{n-1}}{4} + a_{n-1}\right)$
$= 5a_{n-1}$ $(\forall n \ge 3)$

So $\forall n \geq 3$, $|Y_n| \geq \frac{1}{2} a_n$ iff $|Z_n| = a_n$

even if $Z_1 = -a_1, \dots, Z_{n-1} = -a_{n-1}$,
if $Z_n = a_n$, then $|Y_n| = \frac{3}{4} a_n \geq \frac{1}{2} a_n$

So $\mathbb{E}|Y_n| \geq \mathbb{E}\, |Y_n| \cdot I_{\{|Y_n| \geq \frac{1}{2} a_n\}} \geq \frac{1}{2} a_n \cdot \mathbb{P}(|Y_n| \geq \frac{1}{2} a_n)$

$$= \frac{1}{2} a_n \cdot \mathbb{P}(|Z_n| = a_n) = \frac{a_n}{2n^2} \to +\infty \quad (n \to \infty)$$

since a_n is growing exponentially fast.

This implies the converse of MG convergence thm is not necessarily true.