e.g: $\{X_n\}$ MC, state space $S=\{1,2,3\}$,

$$P=\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{10} & \frac{7}{10} \end{pmatrix},$$ initial dist

$$\alpha^T=\left(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right),$$

find prob:

(a): $\mathbb{P}(\underbrace{X_1=2}_{\text{cause}}\,|\,\underbrace{X_3=1}_{\text{consequence}})$

Bayes thm:

marginal of X_1: $\alpha^T\cdot P$

marginal of X_3: $\alpha^T\cdot P^3$

$$=\frac{\mathbb{P}(X_3=1\,|\,X_1=2)\cdot\mathbb{P}(X_1=2)}{\mathbb{P}(X_3=1)}$$

$$=\frac{(P^2)_{21}\cdot(\alpha^T P)_2}{(\alpha^T\cdot P^3)_1}$$

$$\text{(b)}: \mathbb{P}(\underbrace{X_5=1}_{\text{present } B} \mid \underbrace{X_1=2, X_2=3}_{\text{past}}, \underbrace{X_7=2}_{\text{future}})$$

$$\mathbb{P}(\text{future} \mid \text{past}, \text{present})$$

$$= \frac{\mathbb{P}(\underbrace{X_5=1, X_7=2}_{BC} \mid \underbrace{X_1=2, X_2=3}_{A})}{\mathbb{P}(\underbrace{X_7=2}_{C} \mid \underbrace{X_1=2, X_2=3}_{A})}$$

$$\underline{\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}}$$

conditional version $\Longrightarrow$ set $A = \{X_1=2, X_2=3\}$

if I set $P_A(\cdot) = P(\cdot|A)$

then

$$\boxed{\mathbb{P}_A(B|C) = \frac{\mathbb{P}_A(BC)}{\mathbb{P}_A(C)}}$$

$\mathbb{P}_A(\cdot) \triangleq \mathbb{P}(\cdot \mid A)$ is still a probability

$$\underset{\text{property}}{\overset{\text{Markov}}{=}} \frac{\mathbb{P}(X_5=1, X_7=2 \mid X_2=3)}{\mathbb{P}(X_7=2 \mid X_2=3)}$$

only the most recent thing happening in the past matters!

$$= \frac{\mathbb{P}(\underbrace{X_5=1}_{\text{present}}, \underbrace{X_7=2}_{\text{future}} \mid \underbrace{X_2=3}_{\text{past}})}{(P^5)_{32}}$$

$$\mathbb{P}(X_7=2 \mid X_2=3, X_5=1) \cdot \mathbb{P}(X_5=1 \mid X_2=3) = \mathbb{P}(X_7=2, X_5=1 \mid X_2=3)$$

$$= \frac{\mathbb{P}(X_5=1 \mid X_2=3) \cdot \mathbb{P}(X_7=2 \mid X_2=3, X_5=1)}{(P^5)_{32}}$$

Markov Property.

$$= \frac{(P^3)_{31} \cdot \mathbb{P}(X_7=2 \mid X_5=1)}{(P^5)_{32}}$$

$$= \frac{(P^3)_{31}\,(P^2)_{12}}{(P^5)_{32}}$$

$\{X_n\}$ MC, $Z_n = (X_n, X_{n+1}) \in \mathbb{R}^2$

2-step history of $\{X_n\}$.

$\Downarrow$

prove: $\{Z_n\}$ is a MC in $\mathbb{R}^2$.

$$\mathbb{P}(Z_{n+1} = i_{n+1} \mid Z_0 = i_0, Z_1 = i_1, \cdots, \underline{Z_n = i_n})$$

$Z_{n+1} \to X_{n+1}, X_{n+2}$; $Z_0 \to X_0, X_1$; $Z_1 \to X_1, X_2$; $Z_n \to X_n, \boxed{X_{n+1}}$ most recent.

$\|$ apply Markov property of (X_n)

$$\mathbb{P}(Z_{n+1} = i_{n+1} \mid Z_n = i_n)$$

e.g.: Continue flipping coin $\begin{cases} \text{head} \triangleq 1 \\ \text{tail} \triangleq 0 \end{cases}$,

what is average number of flips to make until we see two consecutive heads? (6)

<u>Pf</u>: Let X_n be the outcome of the n-th coin flip, so $X_1, X_2, \cdots$ i.i.d.

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = 0) = \frac{1}{2}.$$

$\Rightarrow \{X_n\}$ is MC $\Rightarrow$ $Z_n = (X_n, X_{n+1})$ must also be MC.

Let T be the num of flips until see two consecutive heads

first hitting time

$$T = \inf\{n : Z_n = (1,1)\} + 2$$

want to calculate $\mathbb{E}T$

Initial Dist:

$$\boxed{Z_0 : \underbrace{(0,0),\ (0,1),\ (1,0),\ (1,1)}_{= \text{ w.p. } \frac{1}{4}}}$$

calculate those C.E.!

$$\mathbb{E}T = \mathbb{E}\left[\mathbb{E}(T \mid Z_0)\right] = \frac{1}{4} \times \mathbb{E}\left[T \mid Z_0 = (0,0)\right]$$

$$+ \frac{1}{4} \times \mathbb{E}\left[T \mid Z_0 = (0,1)\right] + \frac{1}{4} \times \mathbb{E}\left[T \mid Z_0 = (1,0)\right]$$

$$+ \frac{1}{4} \times \mathbb{E}\left[T \mid Z_0 = (1,1)\right]$$

Transition prob matrix for $\{Z_n\}$:

$$P = \begin{array}{c|cccc} & (0,0) & (0,1) & (1,0) & (1,1) \\ \hline (0,0) & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ (0,1) & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ (1,0) & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ (1,1) & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$\mathbb{P}(Z_{n+1}=(0,0) \mid Z_n=(0,0))$$

$$= \mathbb{P}(X_{n+1}=0,\ X_{n+2}=0 \mid X_n=0,\ X_{n+1}=0)$$

$$= \frac{1}{2}$$

$$\mathbb{P}(Z_{n+1}=(1,0) \mid Z_n=(0,0))$$

$$= \mathbb{P}(\boxed{X_{n+1}=1},\ X_{n+2}=0 \mid X_n=0,\ \boxed{X_{n+1}=0})$$

contradiction

$$= 0$$

an expectation itself

α_s : average num of flips until see two consecutive heads starting from $Z_0 = s$,

$s \in \mathcal{S} = \{(0,0), (0,1), (1,0), (1,1)\}$

state space of $\{Z_n\}$

waste one time transiting $(0,0) \to (0,0)$

$$\begin{cases} \alpha_{(0,0)} = \frac{1}{2} \times (1 + \alpha_{(0,0)}) + \frac{1}{2} \times (1 + \alpha_{(0,1)}) \\ \alpha_{(0,1)} = \frac{1}{2} \times (1 + \alpha_{(1,0)}) + \frac{1}{2} \times (1 + 2) \\ \alpha_{(1,0)} = \frac{1}{2} \times (1 + \alpha_{(0,0)}) + \frac{1}{2} \times (1 + \alpha_{(0,1)}) \\ \alpha_{(1,1)} = 0 + 2 = 2 \end{cases}$$

(discuss the value of Z_1)

waste one time transiting $(0,0) \to (0,1)$

$\Downarrow$ Solve this linear system

$$\alpha_{(0,0)} = \alpha_{(1,0)}, \begin{cases} \alpha_{(0,1)} = \frac{1}{2} \times (1 + \alpha_{(1,0)}) + \frac{3}{2} \\ \alpha_{(1,0)} = \frac{1}{2} \times (1 + \alpha_{(1,0)}) + \frac{1}{2} \times (1 + \alpha_{(0,1)}) \end{cases}$$

$$\begin{cases} \frac{1}{2}\alpha_{(1,0)} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\alpha_{(0,1)} = 1 + \frac{1}{2}\alpha_{(0,1)} \\ \alpha_{(0,1)} = \frac{1}{2} \times (1 + \alpha_{(1,0)}) + \frac{3}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_{(0,0)} = 8 \\ \alpha_{(0,1)} = 6 \\ \alpha_{(1,0)} = 8 \\ \alpha_{(1,1)} = 2 \end{cases}$$

$$\text{IET} = \frac{1}{4} \cdot \left(\alpha_{(0,0)} + \alpha_{(0,1)} + \alpha_{(1,0)} + \alpha_{(1,1)}\right)$$

$$= \frac{1}{4} \cdot (8 + 6 + 8 + 2) = \boxed{6}$$

e.g.: $\{X_n\}$ is not MC, while $\{Z_n\}$ can be MC.

X_n has state space $\{0,1\}$, and

$$\begin{cases} \text{if } X_{n-1}=0 \text{ and } X_{n-2}=0, & X_n = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases} \\ \text{if } X_{n-1}=0 \text{ and } X_{n-2}=1, & X_n = \begin{cases} 0 & \text{w.p. } \frac{1}{4} \\ 1 & \text{w.p. } \frac{3}{4} \end{cases} \\ \text{if } X_{n-1}=1 \text{ and } X_{n-2}=0, & X_n = \begin{cases} 0 & \text{w.p. } \frac{3}{4} \\ 1 & \text{w.p. } \frac{1}{4} \end{cases} \\ \text{if } X_{n-1}=1 \text{ and } X_{n-2}=1, & X_n = \begin{cases} 0 & \text{w.p. } \frac{1}{3} \\ 1 & \text{w.p. } \frac{2}{3} \end{cases} \end{cases}$$

$$\mathbb{P}(X_n=1 \mid X_{n-2}=0, X_{n-1}=1) = \frac{1}{4}$$

but $\not\equiv$ not necessarily equal

$$\mathbb{P}(X_n=1 \mid X_{n-1}=1) = \frac{1}{4} \times \mathbb{P}(X_{n-2}=0) + \frac{2}{3} \times \mathbb{P}(X_{n-2}=1)$$

Consider $\{Z_n\}$: state space $\{(0,0), (0,1), (1,0), (1,1)\}$,

$$\mathbb{P}(Z_{n+1}=(0,0) \mid Z_n=(0,0))$$

$$= \mathbb{P}(\underline{X_{n+1}=0}, X_{n+2}=0 \mid X_n=0, \underline{X_{n+1}=0})$$

$$= \mathbb{P}(X_{n+2}=0 \mid X_n=0, X_{n+1}=0) = \frac{1}{2}$$

$$\mathbb{P}(Z_{n+1}=(1,0) \mid Z_n=(1,1))$$

$$= \mathbb{P}(\underline{X_{n+1}=1}, X_{n+2}=0 \mid X_n=1, \underline{X_{n+1}=1})$$

$$= \mathbb{P}(X_{n+2}=0 \mid X_n=1, X_{n+1}=1) = \frac{1}{3}$$

So: one-step transition prob of $\{Z_n\}$ is well-defined! $\{Z_n\}$ still MC!

☆: Important since it turns non-Markov process $\{X_n\}$ into Markov process $\{Z_n\}$!

One-step transition matrix for $\{Z_n\}$ in the example above:

$$
\begin{array}{c|cccc}
 & (0,0) & (0,1) & (1,0) & (1,1) \\
\hline
(0,0) & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
(0,1) & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\
(1,0) & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\
(1,1) & 0 & 0 & \frac{1}{3} & \frac{2}{3}
\end{array}
$$