5.3: $S = \{AA, Aa, aa\}$, $\{X_n\}$.

P given, for each $g \in S$, long run prob that decendants have gene pair g.

$(n \to \infty)$

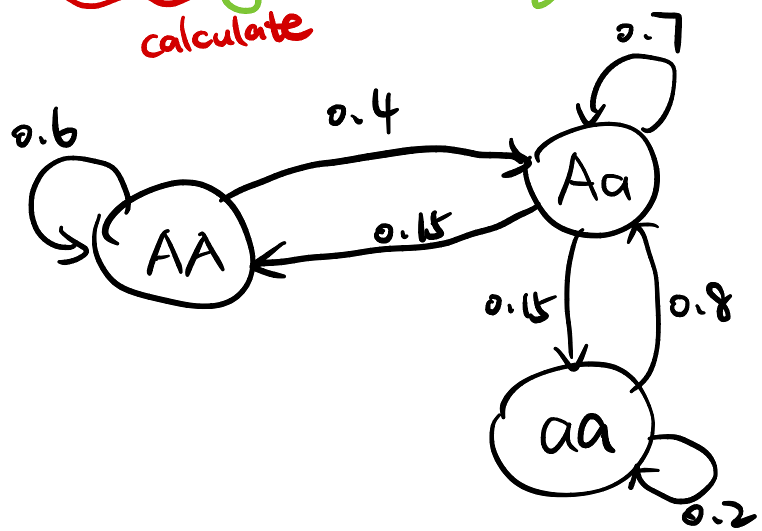
dist of $X_n = \alpha^T \cdot P^n$

initial dist

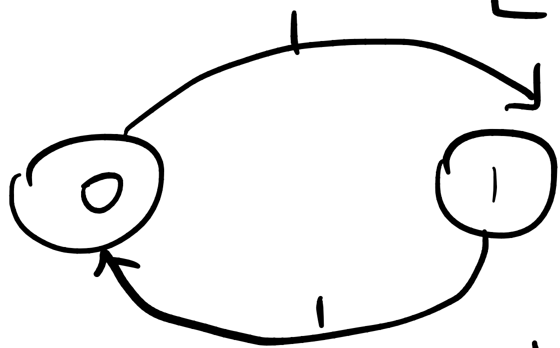
Compute $\boxed{\text{limiting dist}}$

Thm: If a MC is ergodic (irreducible, recurrent, aperiodic), then limiting dist = stationary dist, regardless of the initial dist.

calculate



e.g: $S=\{0,1\}$, $P=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$



stat dist is $\pi = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$

(∃ and unique)

$$\pi^T P = \left(\tfrac{1}{2}\ \ \tfrac{1}{2}\right)\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \left(\tfrac{1}{2}\ \ \tfrac{1}{2}\right) = \pi^T$$

however, limiting dist is not stat dist and does not exist!

(depends on α)

Dist of X_n: $\alpha^T \cdot P^n$

limiting dist, if exists, shall be equal to

$$\alpha^T \cdot \lim_{n\to\infty} P^n$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$P^3 = P \cdot P^2 = P, \quad P^4 = P^2 \cdot P^2 = I$$

$$P^n = \begin{cases} P & \text{if } n \text{ is odd} \\ I & \text{if } n \text{ is even.} \end{cases}$$

e.g.: that stat dist $\exists$ & unique but limiting dist may not exist.

This MC is irreducible, recurrent, but it's not aperiodic,

period of $0 = \gcd\{2, 4, 6, \cdots\} = 2$

period of MC $= 2 \neq 1$

$$\begin{cases} \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & x_0 = 0,\ x_1 = 1,\ x_2 = 0,\ x_3 = 1, \\ & \text{-- -- .} \\ \alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x_0 = 1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 0, \\ & \text{-- -- .} \end{cases}$$

5.5: $\begin{cases} \text{cloudy} - 0 \\ \text{sunny} - 1 \\ \text{rainy} - 2 \end{cases}$ $\quad S = \{0, 1, 2\}$

P given.

$X_0 = 0$ (init dist)

(a): T_0 be the time of next cloudy day

$$\mathbb{E}_0 T_0 = \frac{1}{\pi_0}$$

(π_0 is the component in stat dist corresponding to state 0)

(b): N be the # of cloudy days in the next week (time 1–7)

Way 1: Markov property

$$\mathbb{E}_0 N = \mathbb{E}_0[\mathbb{E}(N \mid X_1)]$$

$$= \mathbb{P}_0(X_1 = 0) \cdot \mathbb{E}_0(N \mid X_1 = 0) +$$

$$\mathbb{P}_0(X_1=1)\cdot\mathbb{E}_0(N|X_1=1)+\mathbb{P}_0(X_1=2)\cdot\mathbb{E}_0(N|X_1=2)$$

$$\mathbb{E}_0(N|X_1=0)=1+\underline{\underline{\mathbb{E}_0(N')}}$$

\# of cloudy days in time 1 – 6

1 2 3 4 5 6 7

$\uparrow$

$X_1=0$

(has 1 contribution N)

$$\mathbb{E}_0(N|X_1=1)=\underline{\underline{\mathbb{E}_1(N')}}$$

$$\mathbb{E}_0(N|X_1=2)=\underline{\underline{\mathbb{E}_2(N')}}$$

Repeat the same procedure for N' to reduce the problem.

Way 2:

$$N = I_{\{X_1=0\}} + I_{\{X_2=0\}} + \cdots + I_{\{X_7=0\}}$$

$$\mathbb{E}N = \mathbb{E}I_{\{X_1=0\}} + \cdots + \mathbb{E}I_{\{X_7=0\}}$$

$$= \underbrace{\mathbb{P}(X_1=0)} + \cdots + \underbrace{\mathbb{P}(X_7=0)}$$

dist of X_k is $\alpha^T P^k$

$$\mathbb{P}(X_k=0) = (\alpha^T P^k)_0$$

for event A,

$$I_A(\omega) = \begin{cases} 0 & \text{if } \omega \notin A \\ 1 & \text{if } \omega \in A \end{cases}$$

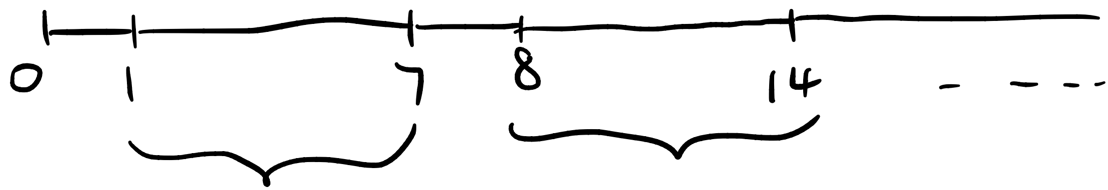
sample point

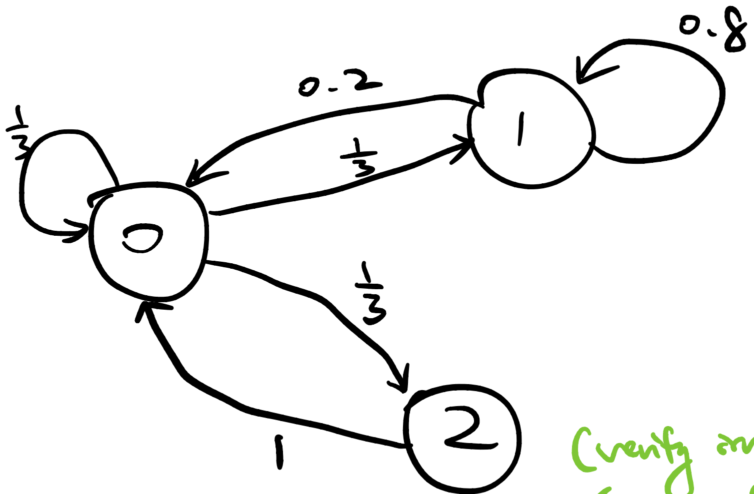
$$\mathbb{E}I_A = 1 \cdot \mathbb{P}(I_A=1) + 0 \cdot \mathbb{P}(I_A=0)$$

$$= 1 \cdot \mathbb{P}(A) = \mathbb{P}(A)$$

$$\mathbb{E}N = \sum_{n=1}^{7} n \cdot \mathbb{P}(N=n) \quad \checkmark$$

(c): Typical # of cloudy days in a randomly selected week.



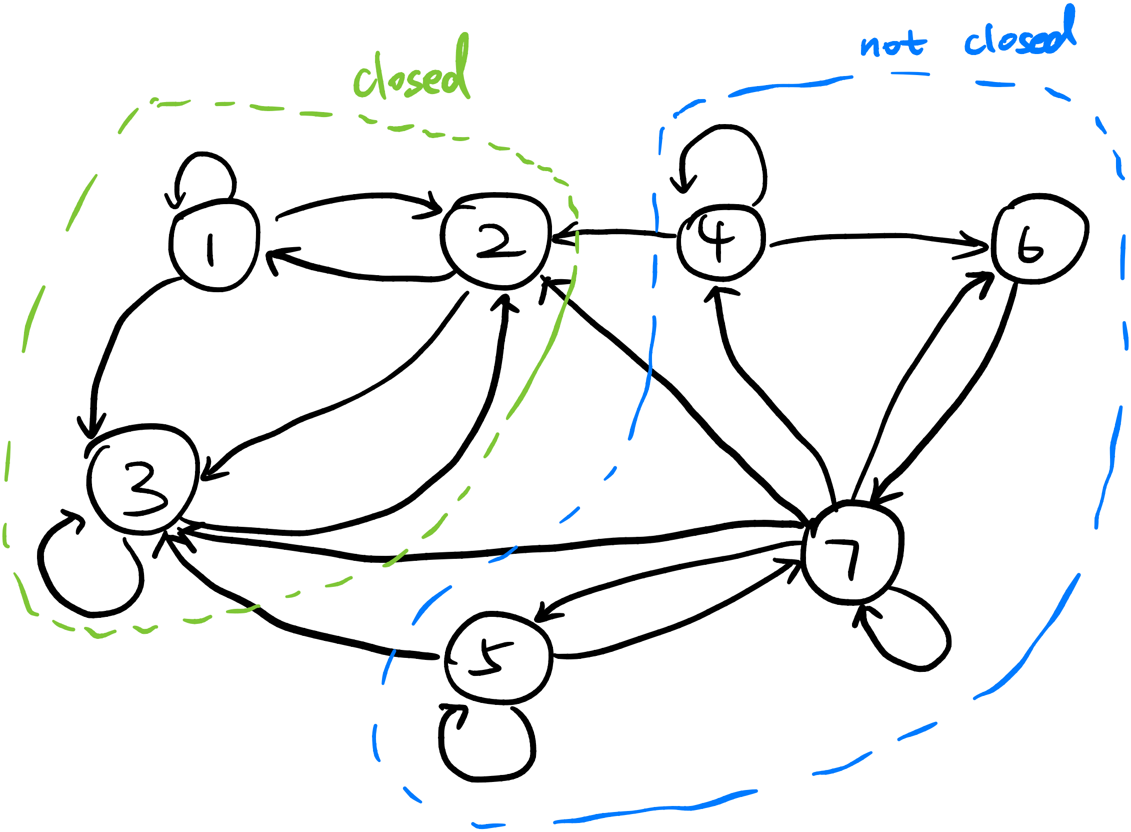


limiting dist = stat dist (verify irreducible, recurrent, aperiodic)

so after long enough time, dist of X_n will converge to π.

So $\mathbb{E}N = \pi_0 + \cdots + \pi_0 = \underline{\underline{7 \cdot \pi_0}}$

5.1:



$\underbrace{1 \leftrightarrow 2 \leftrightarrow 3}_{\text{recurrent}} \qquad \underbrace{4 \leftrightarrow 6 \leftrightarrow 7 \leftrightarrow 5}_{\text{transient}}$

two commu class

recurrent

transient.

Find $\lim_{n\to\infty} P_{ij}^{n} = \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i)$

act as if start MC at state i,
want to find the limiting dist

If we start MC at state 4, after a long enough time transition from blue comm~ class to green comm~ class always happens since there is always a positive prob of happening and state 4 is transient.

If we denote $\lim_{n\to\infty} P_{ij}^{n} = P_{ij}^{\infty}$, then

$\forall i \in$ blue comm~ class, $\forall j \in$ blue comm~ class

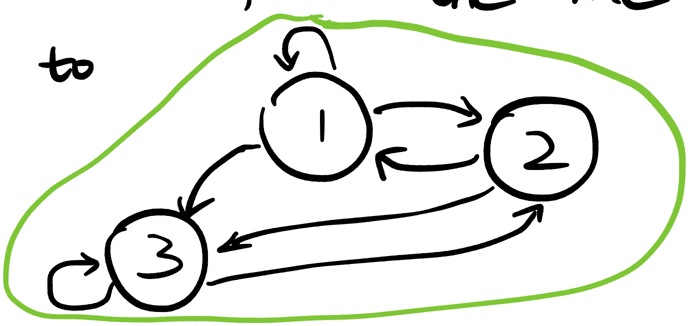
$P_{ij}^{\infty} = 0$

Since green comm is closed,

$\forall i \in$ green, $\forall j \in$ blue, $P^{\infty}_{ij} = 0$.

$$P^{\infty} = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \left[\begin{array}{c|c} \pi & \\ \pi & 0 \\ \pi & \\ \hline \pi & \\ \pi & 0 \\ \pi & \\ \pi & \end{array}\right] \end{array}$$

By Markov property, whenever we reaches green, we act as if we restart the MC from one state in green. So we can forget the blue, so the MC now reduces to



For this new reduced MC, it's

irreducible, recurrent, aperiodic,

it's limiting dist is equal to

stat dist $\pi \in \mathbb{R}^3$.

$$\pi^T \cdot \underbrace{\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{\text{first 3 rows \& first 3 columns of } P.} = \pi^T$$

$$\pi^T P = \pi^T$$

$$\Downarrow$$

$$P^T \pi = \pi$$

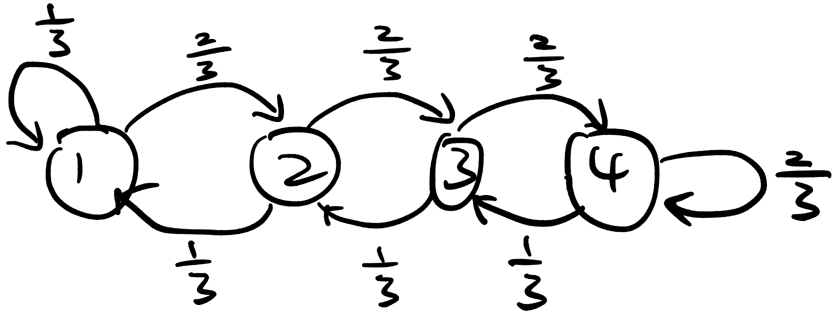
$$\Downarrow$$

π is the eigenvector of P^T corresponding to eigenvalue 1 $\Rightarrow$ normalize π s.t. all entries add up to 1

Thm: $\langle X_n \rangle$ MC, state s is transient, if stat dist π exists, $\pi_s = 0$.

Contrapositive: If stat dist π exists and for state s, $\pi_s > 0$, then s must be recurrent.

5.2:



(c):

Verify irreducible, recurrent, aperiodic so limit dist = stat dist,

expected long-run revenue

$$= \sum_{i=1}^{4} i^2 \cdot \pi_i$$

(b):

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$