4.2:

Conditional distribution of X_2 given $X_1 = 3$

$$\begin{cases} \mathbb{P}(X_2 = 1 \mid X_1 = 3) = (P)_{31} \\ \mathbb{P}(X_2 = 2 \mid X_1 = 3) = \\ \mathbb{P}(X_2 = 3 \mid X_1 = 3) = \end{cases}$$

Cond dist of X_2 given $X_3 = 3$

$$\mathbb{P}(X_2 = 1 \mid X_3 = 3) = \frac{\overbrace{\mathbb{P}(X_3 = 3 \mid X_2 = 1)}^{(P)_{13}} \cdot \overbrace{\mathbb{P}(X_2 = 1)}^{(\alpha^T \cdot P^2)_1}}{\underbrace{\mathbb{P}(X_3 = 3)}_{(\alpha^T \cdot P^3)_3}}$$

Cond dist of X_6 given $X_1 = 3$, $X_4 = 1$, $X_9 = 2$

$$\mathbb{P}(X_6 = 1 \mid X_1 = 3, X_4 = 1, X_9 = 2)$$

$$= \frac{\mathbb{P}(X_6 = 1, X_9 = 2 \mid X_1 = 3, X_4 = 1)}{\mathbb{P}(X_9 = 2 \mid X_1 = 3, X_4 = 1)}$$

$$\boxed{\mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}}$$

$$\overset{\text{Markov}}{\underset{\text{property}}{=}} \frac{\mathbb{P}(X_6=1, X_9=2 \mid X_4=1)}{\mathbb{P}(X_9=2 \mid X_4=1)}$$

$$= \frac{\mathbb{P}(X_9=2 \mid X_4=1, X_6=1) \cdot \mathbb{P}(X_6=1 \mid X_4=1)}{\mathbb{P}(X_9=2 \mid X_4=1)}$$

$$\overset{\text{Markov}}{\underset{\text{property}}{=}} \frac{\mathbb{P}(X_9=2 \mid X_6=1) \cdot \mathbb{P}(X_6=1 \mid X_4=1)}{\mathbb{P}(X_9=2 \mid X_4=1)}$$

4.1:

State space $S = \{$ first-year, sophamore, junior, senior, drop-out, graduate $\}$

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & D & G \\
\hline
1 & 0.03 & 0.91 & 0 & 0 & 0.06 & 0 \\
2 & 0 & 0.03 & 0.91 & 0 & 0.06 & 0 \\
3 & 0 & 0 & 0.03 & 0.93 & 0.04 & 0 \\
4 & 0 & 0 & 0 & 0.03 & 0.04 & 0.93 \\
D & 0 & 0 & 0 & 0 & 1 & 0 \\
G & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

4.3:

(a): $\mathbb{P}(X_3=k)$, dist of X_3 is given by

$$\alpha^T \cdot P^3$$

(b): $$\mathbb{E}X_3 = \sum_{k=1}^{3} k \cdot \mathbb{P}(X_3=k)$$

(c): Given $X_1=2$, compute

$$\mathbb{P}(X_3=k \mid X_1=2)$$

$$= (P^2)_{2k}$$

$$\mathbb{E}(X_3 \mid X_1=2)$$

$$= \sum_{k=1}^{3} k \cdot \mathbb{P}(X_3=k \mid X_1=2)$$

List: (Array)

[1, 2, 3] — 1 dim array (row vector)

1 layer

[[1, 2, 3],
[4, 5, 6],
[7, 8, 9]] — 2 dim array (matrix)

2 layer

[[1],
[2],
[3]] — 2 dim array (column vector)

4.4: Init dist $\alpha^T = (0, 0, 1)$

$\boxed{\alpha^T \cdot P^2}$ $\Rightarrow$ take the longest component, state two years from now

Time stationarity: P does not depend on time.
(time-homogeneous)

If we don't have it, then from time 0 to time 1, transition matrix is P_1, from time 1 to time 2, transition matrix is P_2, etc.

Init dist α^T, dist of X_n:

$$\underbrace{\alpha^T \cdot P_1 \cdot P_2 \cdots \cdot P_n}_{\alpha^T \cdot P^n}$$

if we assume P symmetric,
diagonalization

$$P = Q^T \underset{\uparrow \text{diagonal}}{D} Q \leftarrow \text{orthogonal}$$

$$P^n = \underbrace{Q^T D \overset{I}{\overset{\shortparallel}{\boxed{Q\,Q^T}}} D \overset{I}{\overset{\shortparallel}{\boxed{Q\ -}}}\ -\ \overset{I}{\overset{\shortparallel}{\boxed{-\ Q^T}}} D Q}_{\text{repeat } n \text{ times}}$$

$$= Q^T D^n Q$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}, \quad D^n = \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{bmatrix}$$

4.5:

$$\operatorname{Var} X_3 = \mathbb{E} X_3^2 - (\mathbb{E} X_3)^2$$

$S = \{1, 2, 4\}$

dist of X_3

$$\mathbb{E} X_3 = 1 \times \underline{\mathbb{P}(X_3 = 1)} + 2 \times \underline{\mathbb{P}(X_3 = 2)} + 4 \times \underline{\mathbb{P}(X_3 = 4)}$$

$$\mathbb{E} X_3^2 = 1^2 \times \quad + 2^2 \times \quad + 4^2 \times$$

$$\boxed{\alpha^T \cdot P^3}$$

$X_0 = 2$

$$\alpha^T = (0, 1, 0)$$