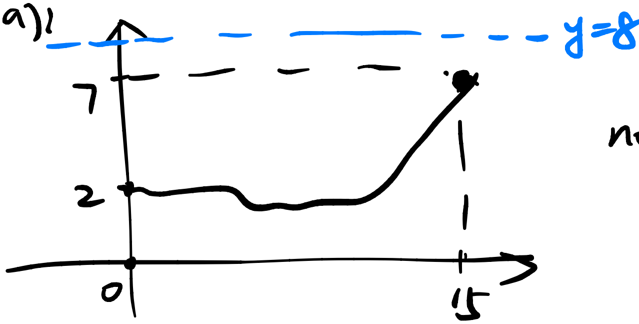
3.3: (a)



not hit $y=8$

$S_0 = 2 < 8$, $S_{15} = 7 < 8$, apply ref principle

reflect $S_{15}=7$ w.r.t. $y=8$ to get $\boxed{S_{15}=9}$

\# of paths that has hit 8

= \# of paths $S_0=2$ to $S_{15}=9$

Total \# of path $S_0=2$ to $S_{15}=7$ is $\binom{15}{10}$

ans: $\binom{15}{10} - \binom{15}{11}$.

Reflection principle:

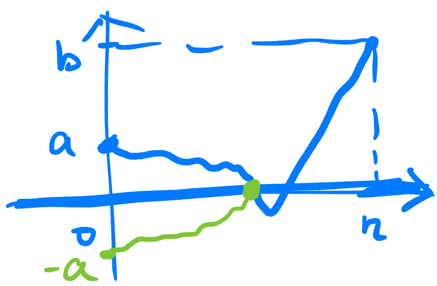
Count \# of paths from $S_0=a$ $(a>0)$ to $S_n=b$ $(b>0)$

that has hit 0

||

\# of paths from $S_0=-a$

to $S_n=b$



<u>3.5</u>: $S_n = S_0 + \sum_{i=1}^{N} P_i$

Wald identity:

N independant of $\{P_i\}$

N: stopping time

If we don't consider bankrupt, then

$$N \sim G(\tfrac{1}{2}).$$

If we consider bankrupt,

N is the first time getting a tail or getting bankrupt.

$$N = \min\{T_1, T_2\}, \quad T_1 \sim G(\tfrac{1}{2})$$

first time getting tail

first time bankrupt

T_1 and T_2 are independent, we can calculate dist of T_2.

$\Downarrow$

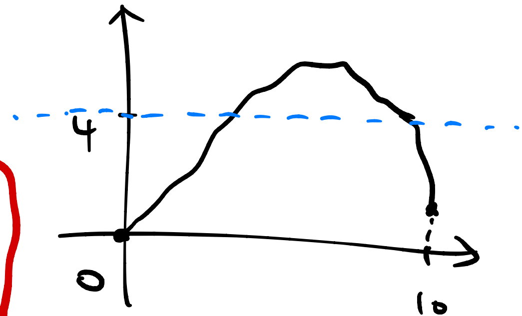
Possible to calc $\mathbb{E}N$

However, Wald's identity does not apply.

3.4:

$$M_n = \max\{S_1, \dots, S_n\} \geq S_n$$

(a): $\mathbb{P}(M_{10} \geq 4)$

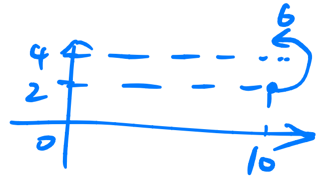


$$
\begin{aligned}
&= \mathbb{P}(M_{10} \geq 4, S_{10} = -10) \\
&\quad + \mathbb{P}(M_{10} \geq 4, S_{10} = -8) \\
&\quad + \cdots \\
&\quad + \mathbb{P}(M_{10} \geq 4, S_{10} = 2) \quad \rightarrow \text{reflection principle} \\
&\quad + \mathbb{P}(M_{10} \geq 4, \underbrace{S_{10} = 4}_{\text{implies}}) \quad \rightarrow \text{trivial} \\
&\quad + \cdots + \mathbb{P}(M_{10} \geq 4, \underbrace{S_{10} = 10}_{\text{implies}})
\end{aligned}
$$

$$\{S_{10} = 10\} = \{S_{10} = 10, M_{10} \geq 4\}$$

$$
\begin{aligned}
&= \mathbb{P}(S_{10} = 18) + \mathbb{P}(S_{10} = 16) + \cdots \\
&\quad + \mathbb{P}(S_{10} = 6) + \mathbb{P}(S_{10} = 4) + \cdots \\
&\qquad + \mathbb{P}(S_{10} = 10) \\
&= \mathbb{P}(S_{10} = 10) + \mathbb{P}(S_{10} = 8) + \mathbb{P}(S_{10} = 6) \\
&\quad + \mathbb{P}(S_{10} = 4) + \mathbb{P}(S_{10} = 6) + \mathbb{P}(S_{10} = 8) + \mathbb{P}(S_{10} = 10)
\end{aligned}
$$

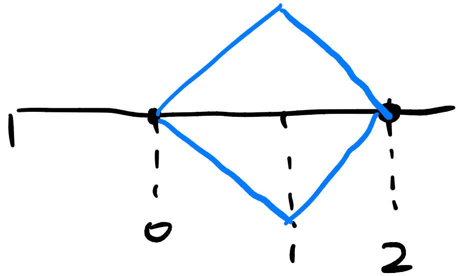
(b): $\mathbb{P}(M_{10} \geq 4 \mid S_{10}=2)$



$$= \frac{\mathbb{P}(M_{10}\geq 4,\ S_{10}=2)}{\mathbb{P}(S_{10}=2)} = \frac{\mathbb{P}(S_{10}=6)\ \checkmark}{\mathbb{P}(S_{10}=2)\ \checkmark}$$

3.1: $S_0=1$, $p=0.6$, $q=0.4$

(a): $\mathbb{P}(S_2=1)$



$= 2p\cdot q$

(b): $\mathbb{P}(S_3=2, S_7=0)$

$$= \underbrace{\mathbb{P}(S_3=2)}_{\checkmark}\cdot \underbrace{\mathbb{P}(S_7=0 \mid S_3=2)}_{\parallel}$$

$$\mathbb{P}(S_7-S_3=-2 \mid S_3=2)$$

independent

$$= \underbrace{\mathbb{P}(S_7-S_3=-2)}_{\checkmark}$$

$\mathbb{P}(S_{10}-S_7=1 \mid S_7=0, S_3=2)$

SRW is Markov chain

(c): $\mathbb{P}(S_3=2, S_7=0, S_{10}=1)$

$$= \mathbb{P}(S_3=2)\cdot \mathbb{P}(S_7=0, S_{10}=1 \mid S_3=2)$$

$$= \underbrace{\mathbb{P}(S_3=2)}_{\checkmark}\cdot \underbrace{\mathbb{P}(S_7=0 \mid S_3=2)}_{\checkmark}\cdot \underbrace{\mathbb{P}(S_{10}=1 \mid S_7=0, S_3=2)}$$

$$\mathbb{P}(S_{10}-S_7=1 \mid S_7=0, S_3=2)$$

$$= \mathbb{P}(S_{10}-S_7=1)$$

3.2:

(a): $$\mathbb{P}(S_2=-2\mid S_7=-3) = \frac{\overset{\boxed{\mathbb{P}(S_5=-1)}}{\overset{\|}{\mathbb{P}(S_7=-3\mid S_2=-2)}}\cdot\mathbb{P}(S_2=-2)}{\mathbb{P}(S_7=-3)}$$

(b): $$\mathbb{P}(S_2\le 0,\ S_4=2,\ S_8=2)$$

$$= \mathbb{P}(S_8=2)\cdot \underbrace{\mathbb{P}(S_8=2\mid S_2\le 0,\ S_4=2)}_{\|}$$

$$\mathbb{P}(\underbrace{S_8-S_4=0\mid S_2\le 0,\ S_4=2}_{\text{indep}})$$

$$=\mathbb{P}(S_8-S_4=0)$$

(c): $$\mathbb{P}(\underbrace{T_2\le 6}_{\substack{\text{first hitting}\\ \text{time of 2}\\ \text{is } \le 6}}) = \mathbb{P}(\underbrace{M_6\ge 2}_{\substack{\text{in time } [0,6],\\ \text{have hit } 2}})$$

first hitting time of 2 is ≤ 6 $\Leftrightarrow$ in time $[0,6]$, have hit 2