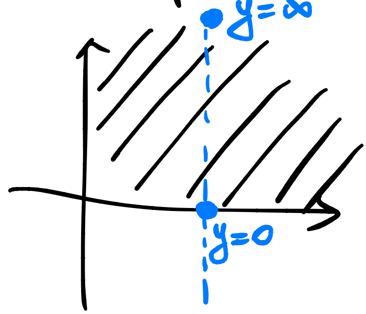
6-sided dice twice, $\mathbb{P}(\text{sum} = 9)$

$$\frac{4}{6 \times 6} = \frac{1}{9}$$

$$\boxed{\begin{matrix} 3+6 \\ 4+5 \\ 5+4 \\ 6+3 \end{matrix}}$$

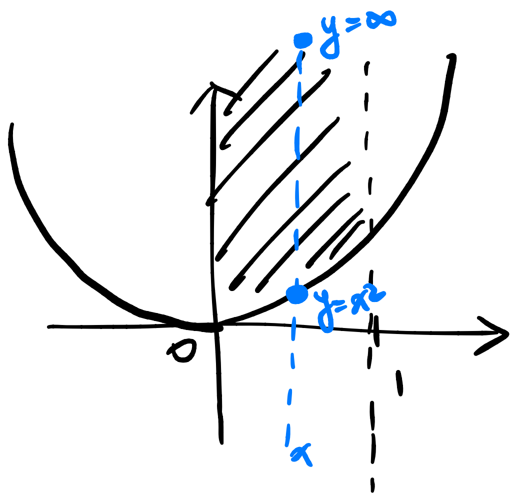
1.1: $f_{X,Y}(x,y) = \begin{cases} 8\, e^{-2x-4y} & 0<x, y<\infty \\ 0 & \text{else} \end{cases}$



(b): $f_X(x) = \int_0^\infty f_{X,Y}(x,y)\, dy$ (fixed x)

$= 2e^{-2x} \quad (x>0)$

1.4



$0<x<1$

$y>x^2$

(fixed x)

$f_X(x) = \int_{x^2}^{\infty} f(x,y)\, dy$

$(0<x<1)$

P1.2: B eggs $\sim \mathcal{P}(8)$, each egg produce off~ w.p. p.

$\underbrace{X_1, \dots, X_B}$

$X_i \sim B(1,p)$ i.i.d.

X_i indicates whether i-th egg produce off~

$$A = X_1 + X_2 + \dots + X_B$$

support of A: $\{0, 1, 2, \dots\}$

$$\mathbb{P}(A=k) = \mathbb{P}(X_1 + \dots + X_B = k)$$

$$= \mathbb{E}\left[\mathbb{P}(X_1 + \dots + X_B = k \mid B)\right]$$

law of iterated expectation

$\mathbb{E}[\mathbb{E}(X \mid Y)] = \mathbb{E}X$

if take $X = I_C$

$\mathbb{E}[\mathbb{P}(C \mid Y)] = \mathbb{P}(C)$

$\boxed{\mathbb{P}(C) = \mathbb{E} I_C}$

$$\mathbb{P}(X_1 + \dots + X_B = k \mid B=b) = \mathbb{P}(\underline{X_1 + \dots + X_b = k} \mid \underline{B=b})$$

independent

$$= \mathbb{P}(\underbrace{\boxed{X_1 + \dots + X_b}}_{\sim B(b,p)} = k) = \binom{b}{k} p^k (1-p)^{b-k}$$

$$\mathbb{P}(X_1+\cdots+X_B=k\mid B)=\binom{B}{k}p^k(1-p)^{B-k}$$

(r.v.)

$$\mathbb{P}(A=k)=\mathbb{E}\qquad=\mathbb{E}\binom{B}{k}p^k(1-p)^{B-k}$$

$$=p^k\cdot\mathbb{E}\binom{B}{k}(1-p)^{B-k}\qquad (B\sim\mathcal{P}(\lambda))$$

$$=p^k\sum_{j=k}^{\infty}\binom{j}{k}(1-p)^{j-k}\cdot\mathbb{P}(B=j)$$

$$=p^k\cdot\sum_{j=k}^{\infty}\binom{j}{k}(1-p)^{j-k}\cdot\frac{\lambda^j}{j!}e^{-\lambda}$$

$$=e^{-\lambda}\cdot p^k\cdot(1-p)^{-k}\sum_{j=k}^{\infty}\boxed{\binom{j}{k}}(1-p)^j\frac{\lambda^j}{\cancel{j!}}$$

$$\binom{j}{k}=\frac{\cancel{j!}}{k!(j-k)!}$$

$$=e^{-\lambda}\frac{p^k}{k!}(1-p)^{-k}\sum_{j=k}^{\infty}\frac{[\lambda(1-p)]^j}{(j-k)!}\qquad \ell=j-k$$

$$=e^{-\lambda}\frac{p^k}{k!}(1-p)^{-k}\sum_{\ell=0}^{\infty}\frac{[\lambda(1-p)]^{\ell+k}}{\ell!}$$

$$=e^{-\lambda}\cdot\frac{p^k}{k!}\cancel{(1-p)^{-k}}\cdot\lambda^k\cancel{(1-p)^k}\cdot\boxed{\sum_{\ell=0}^{\infty}\frac{[\lambda(1-p)]^\ell}{\ell!}}=e^{\lambda(1-p)}$$

$$\mathbb{P}(A=k) = e^{-8}\frac{(8p)^k}{k!}e^{8(1-p)} = \frac{(8p)^k}{k!}\cdot e^{-8p}$$

$(k = 0, 1, 2, \cdots)$

$$\underline{\underline{A \sim \mathcal{P}(8p)}}.$$

(b): $B|_A$ $\Rightarrow$ Bayes

B → cause, A → consequence

$A|_{B=b} \sim B(b, p)$

$$\mathbb{P}(B=b \mid A=a) = \frac{\underline{\mathbb{P}(A=a \mid B=b)}\cdot \underline{\mathbb{P}(B=b)}}{\underline{\mathbb{P}(A=a)}}$$

event A, $I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

$$\mathbb{E} I_A = \mathbb{P}(A)$$

$$\mathbb{E} I_A = 1 \cdot \mathbb{P}(I_A = 1) + \cancel{0 \cdot \mathbb{P}(I_A = 0)}$$

$$= \mathbb{P}(I_A = 1) = \mathbb{P}(\{\omega : \omega \in A\}) = \mathbb{P}(A)$$

1.5:

$$\mathbb{E}X = \mathbb{E}[\mathbb{E}(X|U)] = \mathbb{E}U = \frac{1+10}{2} = 5.5$$

$$\left.\begin{aligned} &X \sim U(0,1) \\ &Y|_X \sim U(0,X) \end{aligned}\right\} \not\Longrightarrow Y \text{ uniform}$$

eg: Throw a dice until you get 6, how many times of 5 to get. (expected)

X: times of 5 before the first 6,

want to calculate $\mathbb{E}X$.

$\mathbb{P}(X=k) = \mathbb{P}(\text{get } k \text{ times of } 5 \text{ before first } 6)$

$$\underbrace{\boxed{k \text{ times of } 5,}}_{h \text{ times}}\ 6$$

↑ first 6

$\Downarrow$

Consider conditioning on Y: num of times to throw before getting first 6.

support of Y: {0, 1, 2, ...}

$\mathbb{P}(Y=k) = \left(\frac{5}{6}\right)^k \cdot \frac{1}{6}$, geometric dist

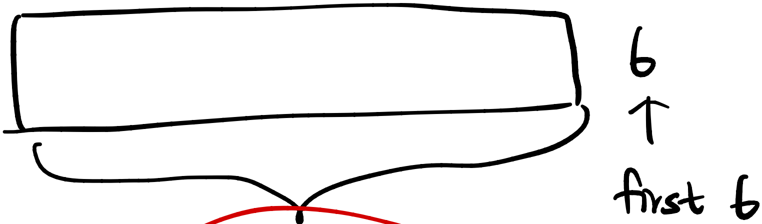
$\mathbb{E}X = \mathbb{E}[\mathbb{E}(X|Y)]$

Calculate $\mathbb{E}(X|Y=y)$:

find dist of $X|_{Y=y}$

$\mathbb{P}(\underline{X=k} \mid Y=y) = \mathbb{P}($ k times of 5 before first 6 | throw y times before first 6 $)$

$k \in \{0, 1, \dots, y\}$



y times

↓

among those,

k times are 5

Among those y times, the number of 5 shall follow $B(y, \frac{1}{5})$

$$\mathbb{P}(X=k \mid Y=y) = \binom{y}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{y-k}$$

$$\underline{\mathbb{E}(X \mid Y=y) = \frac{y}{5}} \Rightarrow \boxed{\mathbb{E}(X \mid Y) = \frac{Y}{5}}$$

$$\mathbb{E}X = \mathbb{E}\frac{Y}{5} = \frac{1}{5} \cdot \mathbb{E}Y$$

$$\mathbb{E}Y = \sum_{k=0}^{\infty} k\cdot\left(\frac{5}{6}\right)^k \cdot \frac{1}{6} = \frac{1}{6}\cdot \sum_{k=0}^{\infty} k\cdot\left(\frac{5}{6}\right)^k$$

$$\left\{\begin{aligned} &\sum_{k=0}^{\infty} p^k = \frac{1}{1-p} \\ &\sum_{k=0}^{\infty} k p^{k-1} = \frac{1}{(1-p)^2} \\ &\sum_{k=0}^{\infty} k\cdot p^k = \frac{p}{(1-p)^2} \end{aligned}\right.$$

$$= \frac{1}{6}\cdot \frac{\frac{5}{6}}{\left(1-\frac{5}{6}\right)^2} = \frac{1}{6} \times \frac{\frac{5}{6}}{\frac{1}{36}} = 5$$

So: $\boxed{\mathbb{E}X = \frac{1}{5}\mathbb{E}Y = 1}$