e.g. Flip a coin, expected number of flips $(\frac{1}{2} H, \frac{1}{2} T)$

needed to get two heads in a row.

→ H H | T H T T - - - - - num —— $= 2$

→ T H H | T T - - - - - - num —— $= 3$

Let X_n denote the outcome of the n-th coin flip. $X_1, X_2, \cdots$ i.i.d.

$$\mathbb{P}(X_1 = H) = \mathbb{P}(X_1 = T) = \frac{1}{2}.$$

Assume Y is the num of flips needed to get two heads in a row.

Want to find $\mathbb{E}Y$. $\mathbb{E}Y = \sum_y y \cdot \boxed{\mathbb{P}(Y=y)}$

↓

hard to figure out.

↓

no specific pattern

$\underline{\underline{Y=4}}$:

T H H | H —— $Y=3$ ✗

H T H H —— $Y=4$ ✓

T T H H —— $Y=4$ ✓

H H | H H —— $Y=2$ ✗

Idea: consider value of X_1

if X_1 is H, then I need another head at time 2 to stop.

if X_1 is T, then I still need to see 2 heads in a row.

Condition on X_1

$\mathbb{E}(X \mid Y=y)$ is a real number (function of y)

$\mathbb{E}(X \mid Y)$ is a random variable (function of Y)

$$\mathbb{E}[\mathbb{E}(X \mid Y=y)] = \mathbb{E}(X \mid Y=y)$$

LIE:

$$\mathbb{E}[\mathbb{E}(X \mid Y)] = \mathbb{E}X$$

$$\mathbb{E}Y = \mathbb{E}[\mathbb{E}(Y|X_1)] \quad (X_1 \text{ is either } H \text{ or } T)$$

$$= \mathbb{E}(Y|X_1=H)\cdot \underbrace{\mathbb{P}(X_1=H)}_{\frac{1}{2}} + \mathbb{E}(Y|X_1=T) \cdot \underbrace{\mathbb{P}(X_1=T)}_{\frac{1}{2}}$$

$$\mathbb{E}(Y|X_1=T) = \underbrace{1}_{\text{spend 1 time getting the tail}} + \underbrace{\mathbb{E}Y}_{}$$

getting a tail contributes nothing to the stopping criteria

law of iterated exp again

$$\mathbb{E}(Y|X_1=H) = \mathbb{E}(Y|X_1=H, X_2=H)\cdot \mathbb{P}(X_2=H|X_1=H) + \mathbb{E}(Y|X_1=H, X_2=T)\cdot \mathbb{P}(X_2=T|X_1=H)$$

if event is independent of condition, we can remove the condition

$$= \mathbb{E}(Y|X_1=H, X_2=H)\cdot \underbrace{\mathbb{P}(X_2=H)}_{\frac{1}{2}}$$

$$+ \mathbb{E}(Y|X_1=H, X_2=T)\cdot \underbrace{\mathbb{P}(X_2=T)}_{\frac{1}{2}}$$

$$\mathbb{E}(Y \mid X_1 = H, X_2 = H) = 2$$

(already 2 consecutive heads, stop immediately)

$$\mathbb{E}(Y \mid X_1 = H, X_2 = T) = \underbrace{2}_{\text{waste 2 units of time seeing } X_1=H,\, X_2=T} + \underbrace{\mathbb{E}Y}_{\text{Since } X_1=H,\, X_2=T, \text{ it contributes nothing to the stopping criterion, still have to work for 2 consecutive heads from the beginning.}}$$

Combine all of them:

$$\mathbb{E}Y = \frac{1}{2} \cdot \left(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (2 + \mathbb{E}Y) \right) + \frac{1}{2} \cdot (1 + \mathbb{E}Y)$$

an equation in $\mathbb{E}Y$, solve it:

$$\mathbb{E}Y = \frac{3}{2} + \frac{3}{4}\mathbb{E}Y, \quad \frac{1}{4}\mathbb{E}Y = \frac{3}{2},$$

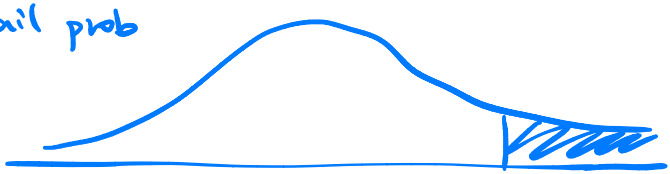
$$\boxed{\mathbb{E}Y = 6}$$

e.g.: (Markov & Chebyshev)

Use Markov & Chebyshev to give two upper bounds on $\mathbb{P}(X > x)$ where $X \sim \mathcal{E}(\lambda)$, calculate those bounds explicitly for $\lambda = 5$, $x = 3$.

Markov: For non-neg r.v. X, $\forall x > 0$,

$$\underbrace{\mathbb{P}(X > x)}_{\text{tail prob}} \le \frac{\mathbb{E}X}{x}$$



Since $X \sim \mathcal{E}(\lambda)$, it's non-neg,

$$\mathbb{P}(X > x) \le \frac{\mathbb{E}X}{x} = \frac{1}{\lambda x} = \frac{1}{5 \times 3} = \frac{1}{15}$$

Chebyshev: $\underbrace{\mathbb{P}(|X - \mathbb{E}X| \ge x)}_{\text{sense of concentration}} \le \frac{\operatorname{Var} X}{x^2}$

(r.v. concentrates near its expectation)

$$\mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \ge a\right) \le \frac{1}{\lambda^2 a^2} \qquad = \frac{1}{5^2(3 - \frac{1}{5})^2} \approx \frac{1}{196}$$

set $a = x - \frac{1}{\lambda}$

$$\mathbb{P}(X > x) \le \mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \ge x - \frac{1}{\lambda}\right) \le \frac{1}{\lambda^2 (x - \frac{1}{\lambda})^2}$$

e.g.: X_1, X_2 independent $\sim G(p)$, $\mathbb{P}(X_1=k)=(1-p)^{k-1}\cdot p$
$(k=1,2,\cdots)$

calculate $\underline{\mathbb{E}(X_1^2 \mid X_1+X_2)}$. $\to$ random variable
(func in X_1+X_2)
$= h(X_1+X_2)$

To calculate $\underline{\mathbb{E}(X_1^2 \mid X_1+X_2=s)}$ first.

h identical.

real number
(func in s) $= h(s)$

Find $X_1 \big|_{X_1+X_2=s}$ (conditional dist), if

$X_1+X_2=s$, since X_1, X_2 take values in

$\{1,2,3,\cdots\}$, the support of $X_1\big|_{X_1+X_2=s}$

is $\{1,2,\cdots,s-1\}$.

<u>Find pmf:</u>

$\forall k \in \{1,2,\cdots,s-1\}$,

$$\mathbb{P}(X_1=k \mid X_1+X_2=s) = \frac{\mathbb{P}(X_1=k,\ X_1+X_2=s)}{\mathbb{P}(X_1+X_2=s)}$$

$$= \frac{\mathbb{P}(X_1=k,\ X_2=s-k)}{\mathbb{P}(X_1+X_2=s)} \overset{\text{indep.}}{=} \frac{\mathbb{P}(X_1=k)\cdot\mathbb{P}(X_2=s-k)}{\mathbb{P}(X_1+X_2=s)}$$

$$= \frac{(1-p)^{k-1} \cdot p \cdot (1-p)^{s-k-1} \cdot p}{\boxed{\mathbb{P}(X_1+X_2=s)}}$$

$\Downarrow$ law of total prob

$$\sum_{k=1}^{s-1} \mathbb{P}(X_1=k) \cdot \mathbb{P}(X_1+X_2=s \mid X_1=k)$$

$$= \sum_{k=1}^{s-1} \mathbb{P}(X_1=k) \cdot \mathbb{P}(X_2=s-k \mid X_1=k)$$

independent

(remove condition)

$$= \sum_{k=1}^{s-1} \mathbb{P}(X_1=k) \cdot \mathbb{P}(X_2=s-k)$$

So: $$\mathbb{P}(X_1=k \mid X_1+X_2=s) = \frac{\cancel{p^2} \cdot \cancel{(1-p)^{s-2}}}{\sum_{j=1}^{s-1} \cancel{p^2} \cancel{(1-p)^{s-2}}}$$

$$= \frac{1}{\sum_{j=1}^{s-1} 1} = \frac{1}{s-1}$$

So: $X_1|_{X_1+X_2=s}$ is uniform on

$\{1, 2, \ldots, s-1\}$

$$\mathbb{E}(X_1^2 \mid X_1+X_2=s) = \sum_{k=1}^{s-1} k^2 \cdot \underbrace{\mathbb{P}(X_1=k \mid X_1+X_2=s)}_{\frac{1}{s-1}}$$

$$= \frac{\sum_{k=1}^{s-1} k^2}{s-1}$$

$$\left(1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{\frac{(\cancel{s-1})(s-1+1)\,[2(s-1)+1]}{6}}{\cancel{s-1}}$$

replace s with X_1+X_2

$$= \frac{s\,(2s-1)}{6}$$

$$\mathbb{E}(X_1^2 \mid X_1+X_2) = \boxed{\frac{(X_1+X_2)\,[2(X_1+X_2)-1]}{6}}$$