Diffusion: $\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)\,dW_t \\ X_0 = x \end{cases}$

generally consider it as SDE on $\mathbb{R}^n$, so X_t take value in $\mathbb{R}^n$, $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with W as m-dim BM.

Thm: (Existence and uniqueness of Solution)

Fix $T>0$, $b:[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma:[0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ restricted to time horizon $[0,T]$ and

$$\begin{cases} \exists\, c>0,\ \forall t\in[0,T],\ \forall x\in\mathbb{R}^n,\ \|b(t,x)\|+\|\sigma(t,x)\| \leq c(1+\|x\|) & \text{growth condition} \\ \exists\, D>0,\ \forall t\in[0,T],\ \forall x,y\in\mathbb{R}^n, \\ \quad \|b(t,x)-b(t,y)\|+\|\sigma(t,x)-\sigma(t,y)\|\leq D\cdot\|x-y\| & \text{Lipschitz} \end{cases}$$

under sense of modification

then SDE has $\boxed{\text{unique}}$ solution in $\mathcal{L}^2([0,T]\times\Omega)$ with cts sample path.

$\mathcal{L}^2([0,T]\times\Omega) \triangleq \{\{X_t\}: \mathbb{E}\int_0^T X_t^2\,dt<\infty\}$ with inner product $\langle X,Y\rangle \triangleq \mathbb{E}\int_0^T X_t Y_t\,dt$

Pf (Sketch):

Existence: Picard iteration
$$\begin{cases} X_t^0 = x \\ X_t^{k+1} = x + \int_0^t b(s, X_s^k)\,ds \\ \qquad\qquad + \int_0^t \sigma(s, X_s^k)\,dW_s \end{cases}$$

expect the limit to be the solution

(first prove Cauchy in $\mathcal{L}^2$ sense, then specify the limit, then check it's solution)

To prove $\{X^k\}$ Cauchy in $\mathcal{L}^2([0,T]\times\Omega)$

$$\|X^m - X^n\| \le \sum_{k=n}^{m-1} \|X^{k+1} - X^k\|$$

($\forall\, m>n$)

$$= \sum_{k=n}^{m-1} \sqrt{\int_0^T \mathbb{E}(X_t^{k+1} - X_t^k)^2\,dt}$$

$$\overset{?}{\longrightarrow} 0 \qquad (m, n \to \infty)$$

Estimate $\mathbb{E}(X_t^{k+1} - X_t^k)^2$

$$= \mathbb{E}\Big(\int_0^t [b(s, X_s^k) - b(s, X_s^{k-1})]\,ds + \int_0^t [\sigma(s, X_s^k) - \sigma(s, X_s^{k-1})]\,dW_s\Big)^2$$

$$\le 2\cdot\mathbb{E}\Big(\int_0^t [b(\cdot\cdot) - b(\cdot\cdot)]\,ds\Big)^2 + \underbrace{2\cdot\mathbb{E}\Big(\int_0^t [\sigma(\cdot\cdot) - \sigma(\cdot\cdot)]\,dW_s\Big)^2}_{= \, 2\cdot\mathbb{E}\int_0^t [\sigma(\cdot\cdot) - \sigma(\cdot\cdot)]^2\,ds}$$

(Itô's isometry)

use Cauchy-Schwarz under $\langle f, g\rangle = \int_0^t f(s)g(s)ds$

$$\left(\int_0^t f(s)\,ds\right)^2 \le \int_0^t f^2(s)ds \cdot \int_0^t ds = t\cdot\int_0^t f^2(s)ds$$

reason we need finite time horizon!

$$\le 2t\cdot\mathbb{E}\int_0^t \underbrace{[b(--)-b(--)]^2}_{\le D\cdot\|X_s^k - X_s^{k-1}\|} ds + 2\,\mathbb{E}\int_0^t \underbrace{[\sigma(--)-\sigma(--)]^2}_{\le D\cdot\|X_s^k - X_s^{k-1}\|} ds$$

$$\le (2D^2T + 2D^2)\int_0^t \mathbb{E}(X_s^k - X_s^{k-1})^2 ds$$

$$\le \text{— — -}$$

$$\le (2D^2T+2D^2)^k \cdot \int\cdots\int_{0<s_k<\cdots<s_1<t} \mathbb{E}(X_{s_k}^1 - X_{s_k}^0)^2\, ds_k\cdots ds_1$$

estimate this!

Estimate (growth condition)

$$\mathbb{E}(X_t^1 - X_t^0)^2 \le (2c^2T^2 + 2c^2T)\cdot(1+\|x\|)^2,$$

bound contains no t.

So combine: $\mathbb{E}(X_t^{k+1} - X_t^k)^2 \le (2D^2T+2D^2)^k \cdot\ \cdot \frac{t^k}{k!}$

proves $\sum_{k=n}^{m-1}\sqrt{\int_0^T \mathbb{E}(X_t^{k+1} - X_t^k)^2 dt} \to 0 \quad (m,n\to\infty)$

(tail of convergent series)

Now identify the limit $X^k \xrightarrow{\mathcal{L}^2([0,T]\times\Omega)} X \quad (k\to\infty)$

to prove it's the solution to SDE, check

$$\begin{cases} \mathbb{E}\left(\int_0^t [b(s,X_s^k) - b(s,X_s)]ds\right)^2 \to 0 \\ \mathbb{E}\left(\int_0^t [\sigma(s,X_s^k) - \sigma(s,X_s)]dW_s\right)^2 \to 0 \end{cases} \quad (k\to\infty) \quad ✓$$

and such X has cts modification since

$$\mathbb{E}\int_0^t \sigma^2(s,X_s)\,ds \overset{\text{growth}}{\le} C^2 \cdot \mathbb{E}\int_0^t (1+|X_s|^2)\,ds \overset{X\in\mathcal{L}^2([0,T]\times\Omega)}{<} \infty$$

proves $\int_0^t \sigma(s,X_s)\,dB_s$ is cts in t.

<u>Uniqueness</u>: Assume $X, \tilde{X}$ are solutions, by same technique,

$$\underline{\mathbb{E}(X_t-\tilde{X}_t)^2} \le (2D^2T+2D^2)\cdot \underline{\int_0^t \mathbb{E}(X_s-\tilde{X}_s)^2\,ds}$$

Grownwall: $\forall t\in[0,T]$, $\mathbb{E}(X_t-\tilde{X}_t)^2 = 0$ so $X_t = \tilde{X}_t$ a.s.

$X, \tilde{X}$ are modification of each other.

#

From now on, only focus on the case where

$\exists$ and uniqueness holds.

Infinitesimal Generator:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)\, dW_t$$

has its solution X_t as cts Markov process but not nece time-homogeneous.

To simplify, first consider the time-homogeneous version

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \\ X_0 = x \end{cases}$$

with solution to be time-homo~ Markov process.

Def: Semi-group operator $P_t f(x) \triangleq \mathbb{E}\big(f(X_t) \,|\, X_0 = x\big) = \mathbb{E}_x f(X_t)$

$$P_{t+s} f(x) = \mathbb{E}_x f(X_{t+s}) = \mathbb{E}_x\left[\mathbb{E}_x\left(f(X_{t+s}) \,\middle|\, \mathcal{F}_t^X\right)\right]$$

$$= \mathbb{E}_x\, \mathbb{E}_{X_t} f(X_s)$$

$$= \mathbb{E}_x\, P_s f(X_t)$$

$$= P_t P_s f(x)$$

So $\underbrace{P_{t+s} = P_t P_s = P_s P_t}_{\text{Commutative}}$, with $f = I_B$, it's $\mathbb{P}_x(X_t \in B)$, the transition kernel.

Def: Infinitesimal generator $\mathcal{L} \triangleq \lim_{h\to 0} \frac{P_h - id}{h}$

derivative at time 0 of semi-group op

Thm: (Rep of inf gen) For $f \in C_c^2(\mathbb{R}^n)$,

$$\mathcal{L}f = b\cdot\nabla f + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T\nabla^2 f), \text{ i.e.}$$

$$\mathcal{L}f(x) = \sum_{i=1}^n b_i(x)\cdot\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}$$

Pf:

Ito formula

$$P_h f(x) = \mathbb{E}_x \boxed{f(X_h)} = \mathbb{E}_x\Big[f(X_0) + \int_0^h \nabla f(X_s)\cdot dX_s + \frac{1}{2}\sum_{i,j=1}^n\int_0^h \frac{\partial^2 f(X_s)}{\partial x_i\partial x_j}\, d\langle X^i, X^j\rangle_s\Big]$$

stochastic integral

$$= f(x) + \mathbb{E}_x\Big[\int_0^h \nabla f(X_s)\cdot b(X_s)\,ds + \cancel{\int_0^h \nabla f(X_s)\cdot\sigma(X_s)dW_s} + \frac{1}{2}\sum_{i,j=1}^n\int_0^h\frac{\partial^2 f(X_s)}{\partial x_i\partial x_j}\sum_{k=1}^m \sigma_{ik}\sigma_{jk}(X_s)\,ds\Big]$$

$$\begin{cases} dX_t^i = \sim dt + \sigma_{i1}dW_t^1 + \cdots + \sigma_{im}dW_t^m \\ dX_t^j = \sim dt + \sigma_{j1}dW_t^1 + \cdots + \sigma_{jm}dW_t^m \end{cases}$$

So $$\mathcal{L}f(x) = \lim_{h\to 0}\frac{\mathbb{E}_x\int_0^h \nabla f(X_s)\cdot b(X_s)ds + \frac{1}{2}\mathbb{E}_x \text{———}}{h}$$

(IVT)

$$= \lim_{h\to 0}\mathbb{E}_x\Big[\nabla f(X_\xi)\cdot b(X_\xi) + \frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2 f(X_\xi)}{\partial x_i\partial x_j}\sum_{k=1}^m\sigma_{ik}\sigma_{jk}(X_\xi)\Big]$$

for some $\xi\in[0,h]$.

(cts path, DCT)

$$= \nabla f(x)\cdot b(x) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i\partial x_j}(\sigma\sigma^T)_{ij}(x)$$

Example: $X_t = W_t$ is n-dim BM $(m=n)$,

e.g. 1 then $dX_t = dW_t$ with $\begin{cases} b \equiv 0 \\ \sigma \equiv I_n \end{cases}$

So $\mathcal{L} = \frac{1}{2}\mathrm{Tr}(\nabla^2 f) = \frac{1}{2}\Delta$

$\boxed{\text{Laplacian} \Updownarrow \text{BM}}$

e.g.: Graph of BM: $X_t = (t, W_t) \in \mathbb{R}^2$ where W_t is 1-dim BM. Then X_t is a diffusion

$$\begin{cases} dX_t^1 = dt \\ dX_t^2 = dW_t \end{cases}$$

, written in matrix form

$$dX_t = \begin{bmatrix} dX_t^1 \\ dX_t^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dW_t, \quad b \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \sigma \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so consider $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(t,x) \longmapsto f(t,x)$,

$$\mathcal{L} f(t,x) = \partial_t f(t,x) + \frac{1}{2}\partial_{xx} f(t,x),$$

$$\mathcal{L} = \partial_t + \frac{1}{2}\partial_{xx}$$

$\boxed{\text{heat operator} \Updownarrow \text{graph of BM}}$

e.g: If $\mathcal{L} = \frac{1+x^2}{2}\partial_{xx}$ $\Rightarrow$ an operator acting on $f: \mathbb{R} \to \mathbb{R}$, construct underlying diffusion

can set $b=0$, $\sigma = \sqrt{1+x^2}$ so

$$\underbrace{dX_t = \sqrt{1+X_t^2}\, dW_t}$$

also can set $b=0$, $\sigma = -\sqrt{1+x^2}$ so

$$\underbrace{dX_t = -\sqrt{1+X_t^2}\, dW_t}$$

can try BM with higher dim $(m=2)$

set $b=0$, $\sigma \in \mathbb{R}^{1\times 2}$, $\sigma = [\sigma_1 \quad \sigma_2]$, $\sigma\sigma^T = \sigma_1^2 + \sigma_2^2$

with $\mathcal{L} = \frac{\sigma_1^2+\sigma_2^2}{2} \cdot \partial_{xx}$, so can set $\begin{cases} \sigma_1 = 1 \\ \sigma_2 = x \end{cases}$

to get diffusion $\underbrace{dX_t = dW_t^1 + X_t\, dW_t^2}$

(just recall existence & uniqueness condition!)

Interpretation:

$$P_t \cdot P_h = P_h \cdot P_t \Longrightarrow \boxed{\frac{dP_t}{dt} = L \cdot P_t}$$

$$\frac{dP_t}{dt} = \lim_{h\to 0} \frac{P_{t+h} - P_t}{h} = P_t \cdot \lim_{h \to 0} \frac{P_h - id}{h} = \boxed{P_t \cdot L}$$

at this point, heuristically, $P_t = e^{Lt}$ provides another connection.

Remark: Rep of inf gen also correct for time-inhomo~ diffusion but

$$\frac{dP_t}{dt} = L \cdot P_t = P_t \cdot L \text{ not hold!}$$

Since $\begin{cases} P_{t+h} \neq P_t \cdot P_h \\ P_{t+h} \neq P_h \cdot P_t \end{cases}$ transition rule depends on time!

Important since

$$f(X_t) = f(x) + \underbrace{\int_0^t \mathcal{L} f(X_s)\, ds}_{\text{collects all terms contributes to expectation \& shows connection with PDE}} + \int_0^t \nabla f(X_s) \sigma(X_s)\, dW_s$$

collects all terms contributes to expectation

&

shows connection with PDE

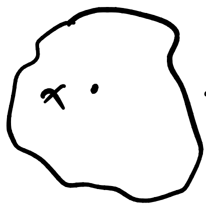
e.g.: What if $f: \mathbb{R}^2 \to \mathbb{R}$ Harmonic, $\Delta f = 0$, we know $\mathcal{L} = \frac{1}{2}\Delta$ if diffusion $\{X_t\}$ is just BM.

Take BM $\{B_t\}$, $B_0 = x$ so

$$\mathbb{E} f(B_t) = f(x) + \mathbb{E}\int_0^t \tfrac{1}{2}\Delta f(B_s)\, ds = f(x)$$

now combine with optional stopping thm in probability that $\mathbb{E} f(B_t) = \mathbb{E} f(B_\tau)$

since $\mathbb{E}\tau < \infty$



τ is first hitting time to any closed loop containing x inside

we get $\mathbb{E} f(B_\tau) = f(x)$

☆ Generalized mean-value property of harmonic func.

just drop BM particle at x, see where it hits closed loop, evaluate f there and take expectation, can recover $f(x)$!

Backward Kolmogorov Equation:

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \qquad \text{diffusion}$$

consider $u(t,x) = \mathbb{E}\left(f(X_T) \mid X_t = x\right)$ with a finite

where does this func come from?

background from value function (control)

time horizon $[0,T]$. How can I describe func u with PDE?

Perturb the time t to $t+h$ for $\forall h>0$, $t+h \le T$

$$\cancel{u(t,x)} = \mathbb{E}\left(\mathbb{E}\left[f(X_T) \mid \mathcal{F}_{t+h}\right] \mid X_t = x\right)$$

$$= \mathbb{E}\left[u(t+h, X_{t+h}) \mid X_t = x\right]$$

$$\overset{\text{Ito}}{=} \mathbb{E}\left[u(t, X_t) + \int_t^{t+h} \partial_t u(s, X_s)\,ds + \int_t^{t+h} \partial_x u(s, X_s)\,dX_s + \frac{1}{2}\int_t^{t+h} \partial_{xx} u(s, X_s)\,d\langle X, X\rangle_s \,\Big|\, X_t = x\right]$$

$$= \cancel{u(t,x)} + \mathbb{E}\left[\int_t^{t+h} (\partial_t + \mathcal{L})u(s, X_s)\,ds + \cancel{\int_t^{t+h} \partial_x u(s, X_s)\,\sigma(s, X_s)\,dW_s} \,\Big|\, X_t = x\right]$$

So $\mathbb{E}\left[\int_t^{t+h} (\partial_t + \mathcal{L})\, u(s, X_s)\,ds \,\Big|\, X_t = x\right] = 0$

So $$\lim_{h\to 0} \frac{\mathbb{E}\left[\int_t^{t+h} (\partial_t + \mathcal{L}) u(s, X_s)\, ds \,\middle|\, X_t = x\right]}{h} = 0$$

$\Downarrow$ IVT, cts path of $\{X_t\}$

$$(\partial_t + \mathcal{L}) u(t, x) = 0$$

So BKE is $$\begin{cases} (\partial_t + \mathcal{L}) u = 0 \\ u(T, x) = f(x) \end{cases}$$

forward SDE

$\Downarrow$

backward PDE

conversely, if BKE is satisfied, u must have the form of conditional expectation.

$\boxed{\text{Consistency !}}$

Forward Kolmogorov Equation (FKE): (skip if no time)

$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$ as diffusion on $\mathbb{R}$.

let $p(t, x)$ denote PDF of X_t, for each given t,
space variable

$$\int p(t,x)\,dx = 1.$$

Observe
$$\mathbb{E}\big(f(X_T) \mid X_0 \sim p_0\big) = \mathbb{E}\Big[\mathbb{E}\big(f(X_T) \mid X_t\big) \mid X_0 \sim p_0\Big]$$
$$= \mathbb{E}\big(u(t, X_t) \mid X_0 \sim p_0\big)$$
$$= \int u(t,x) \cdot p(t,x)\,dx$$

holds for $\forall t \le T$ so take ∂_t to get

$$\partial_t \int u(t,x) \cdot p(t,x)\,dx = 0$$

$$\Downarrow$$

$$\int \partial_t u \cdot p\,dx + \int u \cdot \partial_t p\,dx = 0$$

$\Downarrow$ plug in BKE $(\partial_t + \mathcal{L})u = 0$

$$-\int \mathcal{L}u \cdot p\,dx + \int u \cdot \partial_t p\,dx = 0$$

let $\mathcal{L}^*$ be adjoint of $\mathcal{L}$ under inner product

$$\langle f, g \rangle_{L^2} = \int f(x) \cdot g(x)\,dx$$

Then $\int \mathcal{L}u \cdot p \, dx = \int u \cdot \mathcal{L}^* p \, dx$

So $\int u \cdot (-\mathcal{L}^* + \partial_t) p \, dx = 0$

vary auxiliary func f so that
u traverse through a dense subset
of $\mathcal{L}^2$ function space

$$\Downarrow$$

$$\begin{cases} (\partial_t - \mathcal{L}^*) p = 0 \\ p(0,x) = \mathrm{Law}(X_0) \end{cases}$$

Fokker-Planck, describes
time evolution of
prob meas., forward

To get expression of $\mathcal{L}^*$, in 1-dim,

$$\langle \mathcal{L}f, g \rangle_{\mathcal{L}^2} = \int \left(b \cdot \partial_x f + \frac{\sigma^2}{2} \cdot \partial_{xx} f \right) \cdot g \, dx$$

$$= -\int \partial_x (b \cdot g) \cdot f \, dx - \frac{1}{2} \int \partial_x (\sigma^2 \cdot g) \cdot \partial_x f \, dx$$

$$= -\int \partial_x (b \cdot g) \cdot f \, dx + \frac{1}{2} \int \partial_{xx} (\sigma^2 \cdot g) \cdot f \, dx$$

$$= \int \left[-\partial_x (b \cdot g) + \frac{1}{2} \partial_{xx} (\sigma^2 \cdot g) \right] \cdot f \, dx$$

$\|$ action of $\mathcal{L}^*$

$$= \langle f, \mathcal{L}^* g \rangle_{\mathcal{L}^2}$$

by integration by parts.

Similarly, we can derive $\mathcal{L}^*$ in n-dim that

$$\mathcal{L}^* g = -\mathrm{div}_x (b \cdot g) + \frac{1}{2} \mathrm{Tr}\left[\partial_{xx} (\sigma\sigma^T g)\right]$$

<u>Example</u>: For OU process,

$dX_t = \alpha(m - X_t)dt + \sigma dW_t$, by solving it, we know that $N(m, \frac{\sigma^2}{2\alpha})$ is invariant distribution.

Now we can solve with FKE.

$$\begin{aligned}\mathcal{L}^* p &= -\partial_x (b \cdot p) + \frac{1}{2} \partial_{xx} (\sigma^2 \cdot p) \\ &= \alpha \cdot \left(p - (m - x) \cdot \partial_x p\right) + \frac{\sigma^2}{2} \partial_{xx} p\end{aligned}$$

so $$\underbrace{\alpha p - \alpha (m - x) \partial_x p + \frac{\sigma^2}{2} \partial_{xx} p = \partial_t p}_{\text{Fokker-Planck}}$$

now if $X_0 \sim$ invariant dist, $p(t,x)$ is always the same regardless of t so $p(t,x) = p(x)$ not depend on t.

$$\Downarrow$$

$$\alpha p - \alpha (m - x) \cdot p' + \frac{\sigma^2}{2} p'' = 0$$

solve to get $p(x) = C \cdot e^{-\frac{\alpha (x-m)^2}{\sigma^2}}$, since $\int p(x) dx = 1$, conclude that $p(x)$ is PDF of $N(m, \frac{\sigma^2}{2\alpha})$.

Thm: (Feynman-Kac) Potential

$\rightarrow$ potential $f \in C_c^2$ (running cost in control)

$$\begin{cases} \partial_t u + \mathcal{L}u + \boxed{f} = 0 \\ u(T,x) = \psi(x) \end{cases}$$

solution characterized as

$$u(t,x) = \mathbb{E}\left(\int_t^T f(u,X_u)\,du + \psi(X_T) \,\middle|\, X_t = x\right)$$

(value func of control problem if taken $\inf_\alpha$)

Pf: prob $\Rightarrow$ PDE, same as above, tower property, Ito, IVT.

PDE $\Rightarrow$ prob:

set $Y_s = u(s,X_s) + \int_t^s f(u,X_u)\,du$, Ito formula

$$dY_s = \partial_t u(s,X_s)\,ds + \partial_x u(s,X_s)\,dX_s + \frac{1}{2}\partial_{xx}u(s,X_s)\,d\langle X,X\rangle_s + f(s,X_s)\,ds$$

$$= \partial_t u(s,X_s)\,ds + \mathcal{L}u(s,X_s)\,ds + \partial_x u(s,X_s)\cdot\sigma(s,X_s)\,dW_s + f(s,X_s)\,ds$$

if u sol to PDE,

$$dY_s = \partial_x u(s,X_s)\cdot\sigma(s,X_s)\,dW_s \leftarrow \text{MG}$$

but $Y_t = u(t,X_t)$ so

$\forall t < T$, $\mathbb{E}(Y_T \mid X_t) = Y_t$, write out to see

$$\forall t < T, \quad \mathbb{E}\Big[\underbrace{u(T, X_T)}_{\substack{\| \\ \psi(X_T) \\ \text{(terminal condition)}}} + \int_t^T f(u, X_u)\,du \,\Big|\, X_t\Big] = u(t, X_t)$$

proves the conclusion ✓

Discounted

Thm: (Feynman - Kac)

discount factor!

$$\begin{cases} \partial_t u + \mathcal{L}u - \boxed{V}u + f = 0 \\ u(T, x) = \psi(x) \end{cases} \qquad \begin{cases} V \text{ cts, lower-bdd} \\ f \in C_c^2 \end{cases}$$

$$\Updownarrow$$

$$u(t,x) = \mathbb{E}\Big[\int_t^T e^{-\int_t^r V(s, X_s)\,ds} f(r, X_r)\,dr + e^{-\int_t^T V(s, X_s)\,ds}\,\psi(X_T) \,\Big|\, X_t = x\Big]$$

Pf:

$$Y_s = \int_t^s e^{-\int_t^r V(p, X_p)\,dp} \cdot f(r, X_r)\,dr + e^{-\int_t^s V(r, X_r)\,dr} \cdot u(s, X_s)$$

remaining exactly the same.

Example: Consider PDE for $u(t,x)$

$$\begin{cases} \partial_t u = \frac{\beta^2 x^2}{2} \partial_{xx} u + \alpha x \partial_x u \\ u(0,x) = f(x) \Leftarrow \text{initial cond, not terminal!} \end{cases}$$

First step: turn into terminal condition

$$\tilde{u}(\tau, x) = u(T-t, x) \quad , \quad \tau \triangleq T-t$$

so $\begin{cases} \partial_\tau \tilde{u} = -\partial_t u \\ \mathcal{L} u = \mathcal{L} \tilde{u} \end{cases}$, PDE is the same as

$$\begin{cases} \partial_\tau \tilde{u} + \underbrace{\alpha x \partial_x \tilde{u} + \frac{\beta^2 x^2}{2} \partial_{xx} \tilde{u}}_{\mathcal{L}\tilde{u} \text{ form!}} = 0 \\ \tilde{u}(T, x) = f(x) \end{cases}$$

find inf gen $\mathcal{L}$ s.t. $\mathcal{L}\tilde{u} = \alpha x \partial_x \tilde{u} + \frac{\beta^2 x^2}{2} \partial_{xx} \tilde{u}$

$\begin{cases} b(x) = \alpha x \\ \sigma(x) = \beta x \end{cases}$, so the diffusion is

$$dX_t = \alpha X_t \, dt + \beta X_t \, dW_t$$

by Feynman-Kac, $\tilde{u}$ has prob rep

$$\tilde{u}(\tau, x) = \mathbb{E}[f(X_T) | X_\tau = x]$$

so $u(t,x) = \mathbb{E}[f(X_T) | X_{T-t} = x]$

generally we can simulate SDE!

Here we can solve this SDE

$$\begin{cases} dX_t = \alpha X_t\, dt + \beta X_t\, dW_t \quad \text{(GBM)} \\ X_{T-t} = x \end{cases}$$

$$\Downarrow$$

$$X_T = x \cdot e^{(\alpha - \frac{\beta^2}{2})t + \beta \boxed{(W_T - W_{T-t})}}$$

$$\overset{d}{=} \tilde{W}_t \sim N(0,t)$$

so $u(t,x) = \mathbb{E}\, f\left(x \cdot e^{(\alpha-\frac{\beta^2}{2})t + \beta \tilde{W}_t}\right)$

$$= \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi t}} \cdot e^{-\frac{y^2}{2t}}}_{N(0,t)\ \text{PDF}} \cdot f\left(x \cdot e^{(\alpha-\frac{\beta^2}{2})t + \beta y}\right) dy$$

integral rep~ of the solution.

Example: $u(t,x)$ where $x \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u = \rho u + \frac{1}{2}\Delta u \\ u(0,x) = f(x) \end{cases}$$

turn into terminal condition $\begin{cases} \tau \triangleq T-t, \\ \tilde{u}(\tau, x) = u(T-\tau, x) \end{cases}$

discount part

so $$\begin{cases} \partial_\tau \tilde{u} + \boxed{\rho \tilde{u}} + \frac{1}{2}\Delta \tilde{u} = 0 \\ \tilde{u}(T, x) = f(x) \end{cases}$$

so construct $\mathcal{L}\tilde{u} = \frac{1}{2}\Delta \tilde{u}$ (obviously X_t is n-dim BM)

with $V = \rho$ as constant discount factor.

By Feynman-Kac,

$$\tilde{u}(\tau, x) = \mathbb{E}\left[e^{\rho(T-\tau)} f(X_T) \mid X_\tau = x\right]$$

so $$u(t,x) = \mathbb{E}\left[e^{\rho t} \cdot f(X_T) \mid X_{T-t} = x\right]$$

with $X_t = W_t$ as n-dim BM

Condition on $W_{T-t} = x$, $W_T \overset{d}{=} x + W_{T-t}$, $x + W_t \sim N(x, tI_n)$

Here there is integral rep

$$u(t,x) = e^{\rho t} \cdot \int_{\mathbb{R}^n} (2\pi t)^{-\frac{n}{2}} \cdot f(y) \cdot \underbrace{e^{-\frac{\|y-x\|_2^2}{2t}}}_{\text{PDF}} dy$$

General overview (won't cover details)

$$\begin{tikzcd}
\text{Motivation of those constructions and correspondence} \arrow[dr, "\text{generalize}"] & & \\
& \text{stochastic control problem} \arrow[dl] \arrow[dr] & \\
\text{HJB} \arrow[uu, "\text{special case}"] & & \text{FBSDE} \\
& & \text{BSDE} \arrow[u] \arrow[dl, Leftrightarrow, "\text{correspondence}"] \\
& \text{semi-linear PDE} \arrow[uul, "\text{special case}"] &
\end{tikzcd}$$

numerically solvable (deep learning)

(generalized Feynman-Kac)

typically those look like

$$\partial_t v(t,x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T(t,x)\cdot\partial_{xx} v(t,x)\right) + \inf_{\alpha}\left\{b(t,x,\alpha)\cdot\partial_x v(t,x) + f(t,x,\alpha)\right\} = 0$$

(uncontrolled volatility)