Mean-field approximation of One-period game:

(no time evolution of state)

Setting

- N players
- state space E
- action space A (compact)

  $\alpha \in A$, $\alpha = (\alpha^1, \cdots, \alpha^N)$

- emp measure $\bar{\mu}^N_\alpha \in \mathcal{P}(A)$ (compact)

  $\mathcal{P}(A)$ equipped with topology of weak convergence, induced by metric ρ

- cost functional $J^{N,i} : A^N \to \mathbb{R}$.

  cost of player i when there's all together N players.

LSCF assumption: (large symmetric cost functional)

$$\forall N \geq 1, \quad \exists\, J^N : A^N \to \mathbb{R},$$

$$\begin{cases} \forall N \geq 1,\ (\alpha^1, \dots, \alpha^N),\ J^{N,i}(\alpha^1, \dots, \alpha^N) = J^N(\alpha^i, \alpha^{-i}) \\[2ex] \sup\limits_{N} \sup\limits_{(\alpha^1, \dots, \alpha^N)} \left| J^N(\alpha^1, \dots, \alpha^N) \right| < \infty \\[2ex] \exists\, C > 0,\ \forall N \geq 1,\ \forall (\alpha^1, \dots, \alpha^N), (\beta^1, \dots, \beta^N), \\ \left| J^N(\alpha) - J^N(\beta) \right| \leq C \cdot \left[d_A(\alpha^1, \beta^1) + \rho\left(\bar{\mu}^{N-1}_{\alpha^{-1}}, \bar{\mu}^{N-1}_{\beta^{-1}} \right) \right] \end{cases}$$

permutation invariant

unif bdd

point: view one of the players separately from other players

representative player

other players affect rep player only through the empirical measure

Prop 1-4: Under LSCF assumption, $\hat{\alpha}^N$ is NE for cost functional $J^{N,1}, \dots, J^{N,N}$ and $\exists C>0, \forall N \geq 1, \forall \alpha \in A, \mu \in \mathcal{P}(A)$,

$$\rho\left(\mu, \frac{N-1}{N}\mu + \frac{1}{N}\delta_\alpha\right) \leq \frac{C}{N},$$

limiting cost func

then $\exists$ subseq (N_k), $J : A \times \mathcal{P}(A) \to \mathbb{R}$ cts s.t.

replace emp meas. with $\hat{\mu}$

limiting measure

$$\bar{\mu}^{N_k}_{\hat{\alpha}^{N_k}} \xrightarrow{w} \hat{\mu} \in \mathcal{P}(A) \quad (k \to \infty),$$

(from Prokhorov's thm, tightness)

$$\lim_{k\to\infty} \sup_{\alpha^{N_k} \in A^{N_k}} \left| J^{N_k}(\alpha^{N_k,1}, \dots, \alpha^{N_k,N_k}) - J\left(\alpha^{N_k,1}, \bar{\mu}^{N_k-1}_{\alpha^{N_k,-1}}\right)\right| = 0$$

J^{N_k} can be uniformly well approximated in a way that other players affect player 1 only through emp meas.

and

$$\int_A J(\alpha, \hat{\mu})\, \hat{\mu}(d\alpha) = \inf_{\mu \in \mathcal{P}(A)} \int_A J(\alpha, \hat{\mu})\, \mu(d\alpha)$$

Interpretation leave to later

Pf:

minimal cost for rep player when taking control α and all other players form empirical measure μ

$$J^{(N)}(\alpha,\mu) = \inf_{(\alpha^2,\cdots,\alpha^N)} \Big[\underbrace{J^N(\alpha,\alpha^2,\cdots,\alpha^N)}_{\text{cost functional (optimality)}} + \underbrace{c\cdot \rho\big(\bar\mu^{N-1}_{(\alpha^2,\cdots,\alpha^N)},\mu\big)}_{\text{receive penalty}} \Big]$$

the emp meas. shall look like this

receive penalty if μ is very different from the true empirical measure (consistency, relaxed) $\longrightarrow$ $\boxed{\text{trade-off}}$

when μ already has the legal form of emp meas, there's no penalty

$$\forall i,\ \forall \alpha,\quad J^{(N)}(\alpha, \bar\mu^{N-1}_{\alpha^{N,-i}}) = J^N(\alpha, \alpha^{N,-i}) \qquad (*)$$

only prove for $i=1$ (permutation invariant):

since

$$J^{(N)}\left(\alpha, \bar\mu^{N-1}_{\alpha^{N,-1}}\right) \le J^N(\alpha, \alpha^{N,-1}) + c\cdot\rho\left(\bar\mu^{N-1}_{\alpha^{N,-1}}, \bar\mu^{N-1}_{\alpha^{N,-1}}\right) = J^N(\alpha,\alpha^{N,-1})$$

and if $J^{(N)}(\alpha, \bar\mu^{N-1}_{\alpha^{N,-1}}) < J^N(\alpha, \alpha^{N,-1})$,

$\exists\ \beta^2,\cdots,\beta^N$ s.t.

$$J^N(\alpha, \underbrace{\beta^2,\cdots,\beta^N}_{\text{induces}}) + c\cdot\rho\left(\bar\mu^{N-1}_{\beta^{N,-1}}, \bar\mu^{N-1}_{\alpha^{N,-1}}\right) < J^N(\alpha, \underbrace{\alpha^{N,-1}}_{\text{induces}})$$

Contradicts $\boxed{\text{LSCF}}$ last condition

$$\begin{cases} J^{(N)} \text{ unif bounded} \\ J^{(N)} \quad c\text{-Lips in } \alpha \\ J^{(N)} \quad c\text{-Lips in } \mu \end{cases}$$

check ⟸ LSCF

$$\Downarrow$$

$J^{(N)}$ unif bdd, $J^{(N)}$ unif equicts.

$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall N,$
$\forall\ (\alpha, \mu), (\beta, \mu') \in A \times \mathcal{P}(A),$
if $\left| d_A(\alpha, \beta) + \rho(\mu, \mu') \right| < \delta,$
then $\left| J^{(N)}(\alpha, \mu) - J^{(N)}(\beta, \mu') \right| < \varepsilon.$

$A, \mathcal{P}(A)$ compact

$$\Big\Downarrow \text{ Arzela-Ascoli}$$

$$\exists\ J^{(N_k)} \rightrightarrows J \quad (k \to \infty)$$

$$\Downarrow$$

$$\sup_{\alpha^{N_k} \in A^{N_k}} \left| \underbrace{J^{N_k}(\alpha^{N_k,1}, \ldots, \alpha^{N_k,N_k})}_{\parallel\ J^{(N_k)}\left(\alpha^{N_k,1},\ \bar{\mu}^{N_k-1}_{\alpha^{N_k,-1}}\right)} - J\left(\alpha^{N_k,1}, \bar{\mu}^{N_k-1}_{\alpha^{N_k,-1}}\right) \right| \to 0 \quad (k \to \infty)$$

(proved last page)

when $k \to \infty$, by unif convergence, close to

To prove the last equality,

$$\forall \mu, \int_A J^{(N)}(\alpha, \bar{\mu}^{N-1}_{\hat{\alpha}^{N,-i}})\,\mu(d\alpha) \overset{(*)}{=} \int_A J^N(\alpha, \hat{\alpha}^{N,-i})\,\mu(d\alpha)$$

$$\underset{\text{NE}}{\geq} J^N(\hat{\alpha}^N) = J^N(\hat{\alpha}^{N,i}, \hat{\alpha}^{N,-i}) \underset{(*)}{=} J^{(N)}(\hat{\alpha}^{N,i}, \bar{\mu}^{N-1}_{\hat{\alpha}^{N,-i}})$$

NE (given all others at NE, no motivation of deviation)

with equality if $\mu = \delta_{\hat{\alpha}^{N,i}}$

(player i also take his NE control)

so

$$\forall i, \ \delta_{\hat{\alpha}^{N,i}} \in \arg\inf_{\mu \in \mathcal{P}(A)} \int_A J^{(N)}(\alpha, \bar{\mu}^{N-1}_{\hat{\alpha}^{N,-i}})\,\mu(d\alpha)$$

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interpretation of $\rho(\mu, \frac{N-1}{N}\mu + \frac{1}{N}\delta_\alpha) \le \frac{c}{N}$?

Condition on ρ:

emp meas all but player i follow NE

emp meas everyone follows NE

$$\forall i, \ \rho\left(\bar{\mu}^{N-1}_{\hat{\alpha}^{N,-i}}, \bar{\mu}^{N}_{\hat{\alpha}^{N}}\right) \le \frac{c}{N}$$

$$\bar{\mu}^{N}_{\hat{\alpha}^{N}} = \boxed{\frac{N-1}{N}\cdot \bar{\mu}^{N-1}_{\hat{\alpha}^{N,-i}} + \frac{1}{N}\delta_{\hat{\alpha}^{N,i}}}$$

*: means any single player's deviation from NE changes emp measure small enough under ρ

So:

$$\left| J^{(N)}(\alpha, \bar{\mu}^{N-1}_{\hat{\alpha}^{N},-i}) - J^{(N)}(\alpha, \bar{\mu}^{N}_{\hat{\alpha}^{N}}) \right| \le \frac{c'}{N}$$

$(J^{(N)}$ c-Lips in $\mu)$

$$\Downarrow$$

$$J^{(N)}(\alpha, \bar{\mu}^{N}_{\hat{\alpha}^{N}}) \ge J^{(N)}(\alpha, \bar{\mu}^{N-1}_{\hat{\alpha}^{N},-i}) - \frac{c'}{N}$$

$\Downarrow$ $\int_A \cdots \mu(d\alpha)$ and take $\inf_{\mu \in \mathcal{P}(A)}$

$$\inf_{\mu \in \mathcal{P}(A)} \int_A J^{(N)}(\alpha, \bar{\mu}^{N}_{\hat{\alpha}^{N}}) \mu(d\alpha) \ge \inf_{\mu \in \mathcal{P}(A)} \int_A J^{(N)}(\alpha, \bar{\mu}^{N-1}_{\hat{\alpha}^{N},-i}) \mu(d\alpha) - \frac{c'}{N}$$

$\| $ proved last page

$J^{(N)}(\hat{\alpha}^{N,i}, \bar{\mu}^{N-1}_{\hat{\alpha}^{N},-i})$

$$\Downarrow$$

free of N

$$J^{(N)}(\hat{\alpha}^{N,i}, \bar{\mu}^{N}_{\hat{\alpha}^{N}}) \le \underline{\inf_{\mu \in \mathcal{P}(A)} J^{(N)}(\alpha, \bar{\mu}^{N}_{\hat{\alpha}^{N}}) \mu(d\alpha) + \frac{c'}{N}}$$

$\Downarrow$ $\frac{1}{N}\sum_{i=1}^{N}$ ~~

$$\int_A J^{(N)}(\hat{\alpha}^{N,i}, \bar{\mu}^{N}_{\hat{\alpha}^{N}}) \bar{\mu}^{N}_{\hat{\alpha}^{N}}(d\alpha) \le$$

set $N = N_k$

since $J^{(N_k)} \rightrightarrows J \ (k\to\infty)$, $\exists\, \varepsilon_k > 0$, $\varepsilon_k \to 0 \ (k\to\infty)$,

☆: can change $J^{(N_k)}$ to J with price of ε_k since unif conv!

$$\int_A J(\alpha, \bar{\mu}^{N_k}_{\hat{\alpha}^{N_k}})\, \bar{\mu}^{N_k}_{\hat{\alpha}^{N_k}}(d\alpha) \le \inf_{\mu \in \mathcal{P}(A)} \int_A J(\alpha, \bar{\mu}^{N_k}_{\hat{\alpha}^{N_k}})\, \mu(d\alpha) + \varepsilon_k$$

set $k\to\infty$, notice $\bar{\mu}^{N_k}_{\hat{\alpha}^{N_k}} \xrightarrow{w} \hat{\mu} \ (k\to\infty)$

proves $\displaystyle\int_A J(\alpha, \hat{\mu})\, \hat{\mu}(d\alpha) \le \inf_{\mu\in\mathcal{P}(A)} \int_A J(\alpha, \hat{\mu})\, \mu(d\alpha)$

($\ge$ obvious) ✓

Interpretation of $\int_A J(\alpha,\hat\mu)\,\hat\mu(d\alpha) = \inf_{\mu\in\mathcal{P}(A)} \int_A J(\alpha,\hat\mu)\,\mu(d\alpha)$

take $\alpha_0 \in \arg\inf_\alpha J(\alpha,\hat\mu)$

$\Downarrow$

$$J(\alpha_0,\hat\mu) \overset{\int_A \mu(d\alpha)=1}{=} \inf_{\mu\in\mathcal{P}(A)} \int_A J(\alpha_0,\hat\mu)\,\mu(d\alpha) \overset{\text{def of } \alpha_0}{\le} \inf_{\mu\in\mathcal{P}(A)} \int_A J(\alpha,\hat\mu)\,\mu(d\alpha)$$

$$\overset{\mu=\delta_{\alpha_0}}{\le} J(\alpha_0,\hat\mu)$$

So: $J(\alpha_0,\hat\mu) = \int_A J(\alpha,\hat\mu)\,\hat\mu(d\alpha)$

Now $A_{\hat\mu} \triangleq \{\alpha_0\in A : \alpha_0 = \arg\inf_\alpha J(\alpha,\hat\mu)\}$

collection of all controls minimizing limiting cost func at limiting meas. $\hat\mu$

$$\int_A J(\alpha,\hat\mu)\,\hat\mu(d\alpha) = J(\alpha_0,\hat\mu)\cdot\hat\mu(A_{\hat\mu}) + \int_{A-A_{\hat\mu}} J(\alpha,\hat\mu)\,\hat\mu(d\alpha) \le J(\alpha_0,\hat\mu)\cdot\hat\mu(A_{\hat\mu})$$

so $\hat\mu(A_{\hat\mu}) = 1$

Conversely, if $\hat{\mu}(A_{\hat{\mu}}) = 1$, $\int_A J(\alpha, \hat{\mu})\, \hat{\mu}(d\alpha) = J(\alpha_0, \hat{\mu})$

So: this conclusion $\Leftrightarrow$ $\hat{\mu}(A_{\hat{\mu}}) = 1$

$\text{supp}(\hat{\mu}) \subseteq \arg\inf_{\alpha} \{J(\alpha, \hat{\mu})\}$

$\hat{\mu}$ concentrated on $A_{\hat{\mu}}$.

What this thm wants to tell us?

$(N \to \infty)$ simultaneous

In single-period MFG, replace cost func with J, replace emp meas. with $\hat{\mu}$ and solve:

①: Fix emp meas. as μ,

$$A_\mu = \arg\inf_{\alpha} J(\alpha, \mu) \quad \text{(optimality)}$$

②: Find $\hat{\mu} \in \mathcal{P}(A)$ concentrated on $A_{\hat{\mu}}$.

(consistency)

e.g. of MF approximation

meeting start at t_0 deterministic (all time ≥ 0)

player i has control $\alpha^i = t_i$ planned to arrive at this time

actually, player i arrive at $X^i = \alpha^i + \sigma^i \varepsilon^i$ (state)

where $\varepsilon^1, \varepsilon^2, \ldots \overset{\text{i.i.d.}}{\sim} N(0,1)$,

indep $\sigma^1, \sigma^2, \ldots \overset{\text{i.i.d.}}{\sim} \nu$, ν supp on $(0, +\infty)$

cost func of player i: $(a, b, c > 0)$

arrive later than planned time

arrive later than actual start

arrive earlier than actual start

$$J^i(\alpha) = \mathbb{E}\left[a(X^i - t_0)^+ + b(X^i - t)^+ + c \cdot (t - X^i)^+\right]$$

meeting actually starts at $t = \tau(\bar{\mu}^N_X)$

random

but emp meas. is on state space

where τ maps a measure to a positive real num. (like quantile)

seen as finite player game, hard since coupled through emp meas.

Apply MF approx: (simultaneous use J and μ)

$$\bar{\mu}^N_X \xrightarrow{w} \mu \quad (N\to\infty), \quad \text{replace } t=\tau(\bar{\mu}^N_X) \text{ with } t=\tau(\mu)$$

cost func (limit):

$$J(\alpha,\mu) = \mathbb{E}\left[a(X-t_0)^+ + b(X-t)^+ + c(t-X)^+\right]$$

So: solve stochastic control for representative player with t, J above and

$X=\alpha+\sigma\varepsilon$ as state, α as control,

$\underbrace{\sigma\sim\gamma,\quad \varepsilon\sim N(0,1)}_{\text{indep}}$

①: Fix μ, i.e. fix t

$$A_\mu = \arg\inf_\alpha J(\alpha,\mu)$$

$\downarrow$ weak derivative w.r.t. α

$$\frac{\partial J}{\partial \alpha} = a\cdot \mathbb{P}(\alpha+\sigma\varepsilon - t_0>0) + b\cdot \mathbb{P}(\alpha+\sigma\varepsilon-t>0) - c\cdot \mathbb{P}(-\alpha-\sigma\varepsilon+t>0)$$

denote $Z \triangleq \sigma\varepsilon$, F as CDF of Z,

$F(z) + F(-z) = 1$ (symm dist), $0 < F < 1$, strictly $\uparrow$

diff, with F' unif bdd on $\mathbb{R}$.

$$A_\mu = \{\alpha \geq 0 : \underbrace{a F(\alpha - t_0) + (b+c) F(\alpha - t) = c}_{\text{implicit equation on } \alpha}\}$$

$\Downarrow$ property of F

any fixed μ, implicit eqn gives unique $\alpha \in A_\mu$.

②: Find $\hat{\mu} = \mathcal{L}(\underbrace{\hat{\alpha} + Z}_{\text{state}}) = \underbrace{F(\cdot - \hat{\alpha})}_{\text{induced by this CDF of } \hat{\alpha} + Z}$

when rep player takes $\hat{\alpha}$, limiting meas is $\hat{\mu}$, and have $\underbrace{\hat{\alpha} \in A_{\hat{\mu}}}$

$\Downarrow$

given $\hat{\mu} \Rightarrow t \Rightarrow \hat{\alpha} \in A_{\hat{\mu}}$

$$\begin{cases} a F(\hat{\alpha} - t_0) + (b+c) F(\hat{\alpha} - t) = c \\ t = \tau(F(\cdot - \hat{\alpha})) \end{cases}$$

given $\hat{\alpha} \Rightarrow \hat{\mu} \Rightarrow t$

think about fixed point iteration, solution $(\hat{\alpha}, \hat{\mu})$ can be rep as fixed point of mapping G,

$G: \mathbb{R}_+ \to \mathbb{R}_+$

$$\underset{\text{num}}{\alpha} \mapsto \underset{\text{meas.}}{F(\cdot - \alpha)} \mapsto \underset{\text{num}}{t = \tau(F(\cdot - \alpha))} \mapsto \underset{\text{num}}{\alpha}$$

through

$a F(\alpha - t_0) + (b+c) F(\alpha - t)$
$= c$

guarantee existence & uniqueness of fixed point?

Just prove it's a <u>contraction mapping</u>!

Under condition on τ that

① : $\tau \geq t_0$

② : monotone, $\forall \alpha \geq 0$, if $\mu([0,\alpha]) \leq \mu'([0,\alpha])$, then $\tau(\mu) \geq \tau(\mu')$

③ : sub-additivity $\forall \alpha \geq 0$, $\tau(\mu(\cdot - \alpha)) \leq \tau(\mu) + \alpha$

So $\forall x, y \in \mathbb{R}_+$, $x<y$,

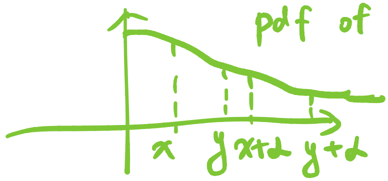
$\downarrow$ monotone τ

$$\tau(F(\cdot - x)) \le \tau(F(\cdot - y))$$

$$\begin{cases} \mu \text{ as } F(\cdot - y) \\ \mu' \text{ as } F(\cdot - x) \\ \mu([0,\alpha]) = \mathbb{P}(0 \le Z - y \le \alpha) \\ \mu'([0,\alpha]) = \mathbb{P}(0 \le Z - x \le \alpha) \end{cases}$$

sub-additivity

pdf of Z



$$\tau(F(\cdot - y)) \le \tau(F(\cdot - x)) + (y - x)$$

$\mu = F(\cdot - x)$, $\alpha = y - x$, $\mu(\cdot - \alpha) = F(\cdot - y)$

Combine:

$$|\tau(F(\cdot - y)) - \tau(F(\cdot - x))| \le |y - x|$$

only care about the last step, given t,

determine α from implicit function.

Set $H(\alpha, t) = a \cdot F(\alpha - t_0) + (b+c) F(\alpha - t) - c$

check $\frac{\partial H}{\partial \alpha} = a F'(\alpha - t_0) + (b+c) F'(\alpha - t) > 0$

$H \in C^1 \Rightarrow$ implicit func thm

$\exists\ \alpha = \alpha(t)$, $\quad \frac{d\alpha(t)}{dt} = \frac{(b+c) F'(\alpha(t) - t)}{a F'(\alpha(t) - t_0) + (b+c) F'(\alpha(t) - t)}$

$$= \frac{1}{\frac{a}{b+c} \cdot \boxed{\frac{F'(\alpha(t) - t_0)}{F'(\alpha(t) - t)}} + 1}$$

lower bound?

so $0 < \frac{d\alpha(t)}{dt} < 1$. $(F > 0)$

If lower bound can be proved,

$\exists\ c \in (0,1)$, $\quad 0 < \frac{d\alpha(t)}{dt} \leq c$

So last step is also contraction.

$\tau(F(\cdot - y))$ $\quad \tau(F(\cdot - x))$

$$|G(x) - G(y)| = |\alpha(t_x) - \alpha(t_y)| \leq c\,|t_y - t_x|$$
$$\leq c\,|y - x|$$

so $\checkmark$.

Important technique, require knowing some frequent practice, e.g.: maps real num to meas to real num, (monotone + sub-additivity)

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e.g.:
$$\begin{cases} R'(t) = \frac{1}{c} \operatorname{Tr}\left[Q'(R(t)) \cdot e^{-t(a+q)L}\right] \cdot e^{-t(a+q)L} \\ R(0) = 0 \\ Q(X) = \left[\det(I + cXL)\right]^{\frac{1}{n}} \end{cases}$$

$$\begin{array}{ccccc} R(t) & \longmapsto & Q'(R(t)) & \longmapsto & \operatorname{Tr}\left[Q'(R(t)) \cdot e^{-t(a+q)L}\right] \\ \text{matrix} & & \text{matrix} & & \text{num} \end{array}$$

$$\begin{array}{ccccc} \longmapsto & R'(t) & \longmapsto & R(t) = \int_0^t R'(s)\,ds \\ & \text{matrix} & & \text{matrix} \end{array}$$
